NETAJI SUBHAS OPEN UNIVERSITY

STUDY MATERIAL

MATHEMATICS
POST GRADUATE

PG (MT) 08
GROUPS A & B

Differential Geometry
•
Graph Theory

PREFACE

In the curricular structure introduced by this University for students of Post-Graduate degree programme, the opportunity to pursue Post-Graduate course in any subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, the study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing, and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in 'invisible teaching'. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials, the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so that they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental—in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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Fifth Reprint : February, 2020

Printed in accordance with the regulations of the Distance Education Bureau of the University Grants Commission.

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NETAJI SUBHAS
OPEN UNIVERSITY

PG (MT) – 08
Differential Geometry
(With the Use of
Tensor Calculus),
Graph Theory

Group A
Differential Geometry

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UNIT : 1 □ TENSORS

§ 1.1 Introduction

The concept of a tensor has its origin in the development of differential geometry by Gauss, Riemann and christoffel. Ricci and his student Levi-civita developed 'tensor calculus' also known as the 'Absolute Differential Calculus'.

The main aim of 'Tensor calculas' is the study of those objects of a space endorsed with a co-ordinate system where the components of objects transform according to a law when we change from one co-ordinate system to another. As a result, the 'Tensor Calculus' has its application to most branches of theoretical physics. The word 'tensor' comes from the word 'tension'.

§ 1.2 Transformation of co-ordinates :

A set of n real numbers $x^1......x^n$, where 1,2,....., n are not the powers of x but are the superscripts of x, is called and n-tuple of real numbers and is denoted by $(x^1......x^n)$ Such a set shall be called a point of an n-dimensional space. The variables are called the coordinates of the point. The set of all such n-tuple of real numbers shall be denoted by S^n. The corresponding co-ordinate system shall be denoted by (x^i).

Let $(\bar{x}^i)$ be another co-ordinate system in S^n which is related to (x^i) by

1.1) $\bar{x}^i = g^i(x^1......x^n)$, i = 1, ..., n

where g^i are the single valued continuous functions of x^1,........., x^n and have continuous partial derivatives of upto any desired order. Equations 1.1) are said to define a transformation of co-ordinates. In order that the transformation be reversible, it is necessary and sufficient that the Jacobian determinant formed by the partial derivatives $\frac{\partial \bar{x}^i}{\partial x^j}$ should not be zero. Under this condition, we can solve 1.1) for the functions of x^i and obtain

1.2) $x^i = h^i(\bar{x}^1, \bar{x}^2, ..., \bar{x}^n)$

We shall refer to a class of co-ordinate tranformations with these properties as admissible transformations.

Example : Consider a system of equations specifying the relation between the spherical polar co-ordinates x^i and the rectangular cartesian co-ordinates y^i in E^3 (3 dimensional Euclidean space)

$$y^1 = x^1 \sin x^2 \cos x^3$$
$$y^2 = x^1 \sin x^2 \sin x^3$$
$$y^3 = x^1 \cos x^2$$

The Jacabian determinant J formed from the partial derivatives is given by

$$J = \begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \frac{\partial y^1}{\partial x^3} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^2}{\partial x^3} \\ \frac{\partial y^3}{\partial x^1} & \frac{\partial y^3}{\partial x^2} & \frac{\partial y^3}{\partial x^3} \end{vmatrix}$$

$$= (x^1)^2 \sin x^2$$

$$\neq 0 \text{ if } x^1 > 0,\ 0 < x^2 < \pi$$

Thus $x^1 = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} \quad , \quad x^1 > 0$

$$x^2 = \cos^{-1} \frac{y^3}{\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}}$$

$$x^3 = \tan^{-1} \frac{y^2}{y^1}$$

Exercise **:** Discuss whether x^1, x^2, x^3 in the transformation between cylindrical polar co-ordinates x^i and the rectangular cartesian co-ordinates y^i in E^3 namely

$$y^1 = x^1 \cos x^3$$
$$y^2 = x^1 \sin x^2$$
$$y^3 = x^3$$

can be expressed in terms of y^1, y^2, y^3

$$\textbf{Answer :} \begin{cases} x^1 = \sqrt{(y^1)^2 + (y^2)^2} & , x^1 > 0 \\ x^2 = \tan^{-1} \dfrac{y^2}{y^1} \\ x^3 = y^3 \end{cases}$$

§ 1.3 Summation Convention :

In writing an expression such as $a_1 x^1 + a_2 x^2 + \ldots\ldots + a_n x^n$

We can use the short notation $\sum_{i=1}^{n} a_i x^i$. An even shorter notation to write it as $a_i x^i$, where we adopt the convention that when ever an indes, appears twice, once as a superscript and once as a subscript, we are to sum over the index form 1 to n, unless otherwise stated. This is known as summation convention and the repeated index is called a dummy index. An index occcuring only once in a given term is called a free index.

From 1.1) We find that

$$d\bar{x}^i = \sum_{j=1}^{n} \frac{\partial \bar{x}^i}{\partial x^j} dx^j$$

which by the Einstein convention can be written as

1.3) $d\bar{x}^i = \dfrac{\partial \bar{x}^i}{\partial x^j} dx^j$

Let us introduce the Kronecker delta, defined by

1.4) $\delta^k_j = 1, \; k = j$

$\qquad = 0, \; k \neq j$

Since the co-ordinates x^i are independent, we can write

1.5) $\delta^k_j = \dfrac{\partial x^k}{\partial x^j}$

Exercises

1. Show that

(i) $\delta^k_j A^j = A^k$ (ii) $\delta^i_i = n$ (iii) $\dfrac{\partial x^k}{\partial \bar{x}^i}\dfrac{\partial \bar{x}^i}{\partial x^j} = \delta^k_j$

Note **:** We shall consider some systems, denoted by e_{ij}, e^{ij}, e_{ijk} and e^{ijk}, called the e-systems and defined as follows :

$$
1.6)\quad \begin{cases} e_{11} = e_{22} = 0, \quad e_{12} = 1, \quad e_{21} = -1 \quad i.e.\ e_{ij} = 0, \quad i = j \\ e^{11} = e^{22} = 0, \quad e^{12} = 1, \quad e^{21} = -1 \qquad i, j = 1, 2 \\ \left.\begin{matrix} e_{123} = e_{231} = e_{312} = 1 \\ e_{213} = e_{321} = e_{132} = 1 \end{matrix}\right\} , \quad \textit{Others being zero} \\ \left.\begin{matrix} e^{123} = e^{231} = e^{312} = 1 \\ e^{213} = e^{321} = e^{132} = -1 \end{matrix}\right\} , \quad \textit{Others being zero} \end{cases}
$$

§ 1.4 Contravariant Vector, Covariant Vector :

A set of n functions A^i of the n coordinates (x^1, x^2, ... x^n) are said to be the components of a contravariant vector if they transform according to the equation

$$1.7)\quad \bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j$$

on change of the co-ordinate system (x^i) to $(\bar{x}^i)$. Also, one gets from above.

$$1.8)\quad \bar{A}^k = \frac{\partial x^k}{\partial \bar{x}^i} \bar{A}^i$$

A set of n functions A_i of the n coordinates (x^1, x^2, ... x^n) are said to be the components of a covariant vector if they transform according to the equation

$$1.9)\quad \bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j$$

on change of coordinate system (x^i) to $(\bar{x}^i)$. Also, it is easy to see that

$$1.10)\quad \bar{A}_k = \frac{\partial \bar{x}^i}{\partial x^k} \bar{A}_i$$

Exercises

1. Show that if A_m is a covariant vector, then $\frac{\partial A_m}{\partial x^s}$ is not a tensor,

2. If f is a scalar function of coordinates (x^i), then show that dx^i is a contravariant vector and $\frac{\partial f}{\partial x^i}$ is a covariant vector.

3. Prove that the law of transformation for contravariant (or covariant) vectors is transitive.

4. Prove that there exists no distinction between contravariant and convariant vectors when we restrict ourselves to transformation of the type.

$$\bar{x}^i = a^i_m x^m + b^i$$

Where b_i are n constants which do not necessarily form the components of a contravariant vector and a^i_m are constants (which do not form a tensor) such that $a^i_m a^i_n = \delta^n_m$

§ 1.5 Invariants :

Any function ψ of the n coordinates $(x^1, x^2, \ldots\ldots x^n)$ is called an invariant or scalar if

1.11) $\psi = \bar{\psi}$

Where $\bar{\psi}$ be its transform on change of the co-ordinate system (x^i) to $(\bar{x}^i)$.

Note that $\frac{\partial \bar{\Psi}}{\partial \bar{x}^r} = \frac{\partial \Psi}{\partial x^s} \frac{\partial x^s}{\partial \bar{x}^r}$

which transforms like 1.9). Thus $\frac{\partial \Psi}{\partial x^k}$ is a covariant vector. Such a vector is sometimes called the gradient of Ψ and is denoted by grad Ψ.

Example : Considering the equations 1.8) and 1.10), it can be shown easily that the expression $(A^m B_m)$ is an invariant.

§ 1.6 Second Order tensors, Higher Order tensors :

A set of n^2 functions A^{ij} of the n co-ordinates $(x^1, \ldots\ldots x^n)$ are said to be the components of a contravariant tensor of order two or of type (2,0) if they transform according to the equation.

1.12) $\overline{A}^{ij} = \frac{\partial \overline{x}^i}{\partial x^m} \cdot \frac{\partial \overline{x}^j}{\partial x^n} A^{mn}$

on change of the co-ordinate system from (x^i) to $(\overline{x}^i)$

From 1.12) on finds that

$\frac{\partial x^r}{\partial \overline{x}^i} \cdot \frac{\partial x^s}{\partial \overline{x}^j} \overline{A}^{ij} = \delta^r_m \delta^s_n A^{mn}$ by Ex. 1 (iii) of § 1.3)

Thus

1.13) $A^{rs} = \overline{A}^{ij} \frac{\partial x^r}{\partial \overline{x}^i} \cdot \frac{\partial x^s}{\partial \overline{x}^j}$

Similarly, a set of n^2 functions A_{ij} of the n co-ordinates $(x^1, \ldots\ldots x^n)$ are said to be the components of a covariant tensor of order 2 or of order (0, 2) if they transform according to the equation.

1.14) $\overline{A}^{ij} = \frac{\partial x^m}{\partial \overline{x}^i} \cdot \frac{\partial x^n}{\partial \overline{x}^j} A_{mn}$

on change of the co-ordinate system from (x^i) to $(\overline{x}^i)$. In a similar way, it can be shown that

1.15) $A_{rs} = \overline{A}^{ij} \frac{\partial \overline{x}^i}{\partial x^r} \cdot \frac{\partial \overline{x}^j}{\partial x^s}$

A set of n^2 functions A^i_j of the n co-ordinates $(x^1, \ldots\ldots x^n)$ are said to be the components of a mixed tensor of order two or of order (1,1) if they transform according to the equation.

1.16) $\overline{A}^i_j = A^m_n \frac{\partial \overline{x}^i}{\partial x^m} \cdot \frac{\partial x^n}{\partial \overline{x}^j}$

On change of the co-ordinate system from (x^i) to $(\overline{x}^i)$. In a similar manner, it can be shown that

1.17) $A^r_s = \overline{A}^i_j \cdot \frac{\partial x^r}{\partial \overline{x}^i} \cdot \frac{\partial \overline{x}^j}{\partial x^s}$

In general, a set of n^{p+q} function $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ of n co-ordinates $(x^1, \ldots\ldots x^n)$ are said to be the components of a mixed tensor of order (p, q) if they transform according to the equation.

$$1.18)\quad \bar{A}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = A^{m_1 m_2 \ldots m_p}_{n_1 n_2 \ldots n_q} \frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \ldots\ldots \frac{\partial \bar{x}^{i_p}}{\partial x^{m_p}} \cdot \frac{\partial x^{n_1}}{\partial \bar{x}^{j_1}} \ldots \frac{\partial x^{n_q}}{\partial \bar{x}^{j_q}}$$

On change of the co-ordinate system from (x^i) to $(\bar{x}^i)$.

Exercises

1. Show that the Kronecker delta is a mixed tensor of order two.
2. If all the components of a tensor in one co-ordinate system are zero at a point, then, show that they are zero at this point in every co-ordinate system.

Note **:** The result stated in Exercise 2 above, enables us to define a zero tensor as follows :

A tensor, whose components are all zero in every co-ordinate,system, is called a zero tensor.

§ 1.7 Algebra of tensors :

If $A^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q}$ and $B^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q}$ are components of two tensors of type (p, q), then $A^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q} \pm B^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_q}$ are the components of another tensor of type (p, q). Such a tensor is called the sum (or difference) of the tensors $A^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q}$ and $B^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q}$ and the algebraic operation by which it is obtained is called the addition (or subtraction) of the tensors.

If ϕ be a scalar, than $\phi A^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q}$ are the components of another tensor of type (p, q). The algebraic operation by which it is obtained is called the multiplication of the tensor by a scalar.

If $A^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q}$ and $B^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q}$ are the components of a tensor of type (p, q) in the same co-ordinate system then they are said to be equal if

$$A^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q} = B^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q}$$

If $A^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ are the components of a tensor of order (p, q), $p \neq 0$, $q \neq 0$ then the quantities obtained by replacing any upper index, say i_m and any lower index, say j_n by the same index i_m and performing summation over i_m are the components of a tensor of type (p–1, q–1). The operation by which it is obtained is called contraction.

If $A^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ and $B^{k_1 \ldots k_r}_{l_1 \ldots l_s}$ are the components of two tensors of order (p, q) and (r, s) respectively, then $A^{i_1 \ldots i_p}_{j_1 \ldots j_q} . B^{k_1 \ldots k_r}_{l_1 \ldots l_s}$ is a tansor of type (p + r, q + s). The algebraic operation by which it is obtained is called the outer multiplication.

By the process of outer mutltiplication of two tensors followed by a contraction, we get a new tensor. This new tensor is called the inner product of the given tensors and the process is called an inner umltiplication.

The algebraic operations on tensors, namely, (i) addition, (ii) subtraction, (iii) scalar multiplication, (iv) contraction and (v) outer multiplication, constitute what is called the tensor Algebra on S^n

Note : All the operations defined in this article, relate to tensors at the same point only.

Exercises

1. Show that the contracted tersor A^m_n is a scalar.
2. Show that the contraction of a tensor of order (2,3) is a tensor of order (1, 2)
3. Prove that the inner product of two tensors A^p_q and B^q_r is a tensor of order (2,1).
4. Show that the tensor equation. $a^m_n \theta_m = \beta\theta_n$ where β is an invariant and θ_m are the arbitrary vector, demands that.

$$a^m_n = \delta^m_n \beta$$

5. If A^i and B_j are the Components of a contravariant and covariant vector then show that their outer product is a tensor of order 2. Is the converse true? Justify your answer.

Solution : 2. Let A^{ij}_{mnp} be the components of a tensor of order (2, 3) in (x^i) system. With respect to change of (x^i) to $(\bar{x}^i)$ system, the given tensor follows the following transformation i.e.

$$\bar{A}^{ij}_{mnp} = A^{uv}_{stw} \frac{\partial \bar{x}^i}{\partial x^u} \cdot \frac{\partial \bar{x}^j}{\partial x^v} \cdot \frac{\partial x^s}{\partial x^m} \frac{\partial x^t}{\partial x^n} \frac{\partial x^w}{\partial \bar{x}^p}$$

To perform the contraction, let us write j = n, then we get

$$\bar{A}^{ij}_{mnp} = A^{uv}_{stw} \frac{\partial \bar{x}^i}{\partial x^u} \cdot \frac{\partial \bar{x}^j}{\partial x^v} \cdot \frac{\partial x^s}{\partial \bar{x}^m} \frac{\partial x^t}{\partial \bar{x}^j} \frac{\partial x^w}{\partial \bar{x}^p}$$

$$= A^{uv}_{stw} \frac{\partial \bar{x}^i}{\partial x^u} \cdot \frac{\partial x^s}{\partial \bar{x}^m} \cdot \frac{\partial x^w}{\partial \bar{x}^p}$$ by Exercies 1 (i, iii) of § 1.3

This is the transformation of a tensor of order (1,2). This completes the solution.

Solution : 5. From the definition of a contravariant vector it can be easily prove that A^iB^j is a tensor of type (2,0) or, a tensor of order 2.

Converse Part : Let us consider two-dimensional Euclidean space E^2. In E^2 let us take a (2,0) tensor A^{ij} defined by

$$A^{ij} = 1, \quad \text{if I} = j$$
$$= 0, \quad \text{if } i \neq j.$$

Let, if possible, there exist two contravariant vectors C^i and D^j such that A^{ij} can written as an outer product of these two vectors, i.e., $A^{ij} = C^iD^j$.

Then $C^1 = A^{11} = 1$ implies $C^1 = 0$

$C^1D^2 = A^{12} = 0$ *implies* $D^2 = 0$, since $C^1 \neq 0$

Again $C^2D^2 = A^{22} = 1$, but since $D^2 = 0$, therefore C^2D^2 must be zero, i.e., $A^{22} = 0$, which is a contradiction.

Hence A^{ij} can not be written as the outer product of two contravariant vectors.

This completes the solution.

§ 1.8 Symmetric and skew Symmetric tensors :

If two contravariant or covariant indices of a tensor can be interchanged without altering the tensor, then it is said to be symmetric in every pait of such indices.

Similarly if by interchanging two contravariant or covariant indices of a tensor, each of its components is altered in sign but not in magnitude, then the tensor is said to be skew-symmetric with repect to these indices.

Exercises

1. Prove that the symmetry (skew symmetry) property remains unchanged by tensor law of transformation.

2. Show that a symmetric tensor of order two has atmost $\frac{1}{2}n(n+1)$ different components.

3. Show that a skew symmetric tensor of order two has $\frac{1}{2}n(n-1)|$ independent components.

4. Show that a tensor of order two is expressible as a sum if two tensors one of which is symmetric and the other is antisymmetric.

5. If $U_{ij} \neq 0$ are components of a tensor of type (0, 2) and if the equation $fu_{ij} + gU_{ji} = 0$ holds then prove that either $f = g$ and U_{ij} is skew-symmetric α $f = -g$ and U_{ij} is symmetric.

6. If A_{ij} is a skew symmetric tensor, prove that

   $(\delta^i_j \delta^k_l + \delta^i_l \delta^k_j) A_{ik} = 0$

7. If the tensors a_{ij} and g_{ij} are symmetric and u^i, v^j are components of contravariant vectors such that

   $(a_{ij} - kg_{ij})u^i = 0 \quad k \neq k'$

   $(a_{ij} - k'g_{ij})v^i = 0$

   the prove that $g_{ij}u^i v^j = 0; \quad a_{ij}u^i v^j = 0$

8. If a tensor A_{ijkl} is symmetric in the first two indices from the left and

skew-symmetric in the second and the fourth indices from the left, show that $A_{ijkl} = 0$..

9. If A_{mn} is a skew-symmetric tensor and B^i is a contravariant vector, then show that $A_{mn}B^mB^n = 0$.

§ 1.9 Quotient Law :

If the result of taking an inner product of a given set of functions with a particular type of tensor of arbitraty components is known to be a tensor, then the given functions will form the components of a tensor.

To explain it, let A (p, q, r) be a set of functions given in the (x^i) system, such that the inner product of A(p, q, r) with an arbitrary tensor B_r^{qs} is a tensor C_p^s | Thus A(p,q,r) $B_r^{qs} = C_p^s$

Suppose in the $(\overline{x}^i)$ system, the above equation is transformed to

$$\overline{A}(u,v,w)\overline{B}_w^{vm} = \overline{C}_u^m$$

As $\overline{B}_w^{vm}$ and $\overline{C}_u^m$ are given to the components of a tensor, using the transformation law, we get

$$\overline{A}(u,v,w)\overline{B}_r^{qs}\frac{\partial\overline{x}^v}{\partial x^q}\cdot\frac{\partial\overline{x}^m}{\partial x^s}\cdot\frac{\partial x^r}{\partial\overline{x}^w} = \overline{C}_p^s\frac{\partial\overline{x}^m}{\partial x^s}\cdot\frac{\partial x^p}{\partial\overline{x}^u}.$$

$$= A(p,q,r)\overline{B}_r^{qs}\frac{\partial\overline{x}^m}{\partial x^s}\cdot\frac{\partial x^p}{\partial\overline{x}^u}$$

from above.

$$\text{or } \overline{B}_r^{qs}\frac{\partial\overline{x}^m}{\partial x^s}\left\{\overline{A}(u,v,w).\frac{\partial\overline{x}^v}{\partial x^q}\cdot\frac{\partial x^r}{\partial\overline{x}^w} - A(p,q,r)\frac{\partial x^p}{\partial\overline{x}^u}\right\} = 0$$

Multiplying with $\frac{\partial x^j}{\partial\overline{x}^m}$ and using Exercise1 (i, iiii) of § 1.3 we get

$$B_r^{qj}\left\{\overline{A}(u,v,w).\frac{\partial\overline{x}^v}{\partial x^q}\cdot\frac{\partial x^r}{\partial\overline{x}^w} - A(p,q,r)\frac{\partial x^p}{\partial\overline{x}^u}\right\} = 0$$

since B_r^{qt} is an arbitrary tensor, we must have

$$\overline{A}(u,v,w).\frac{\partial \overline{x}^v}{\partial x^q}.\frac{\partial x^r}{\partial \overline{x}^w} = A(p,q,r)\frac{\partial x^p}{\partial \overline{x}^u}$$

or $A(t,q,r) = \overline{A}(u,v,w).\frac{\partial \overline{x}^u}{\partial t}.\frac{\partial \overline{x}^v}{\partial x^q}.\frac{\partial x^r}{\partial \overline{x}^w}$

which is the transformation law of a mixed tensor of order (1, 2).

Exercises

1. Assume that $A(p,q)B_{qj} = C_{pj}$ where B_{qj} is an arbitrary tensor and C_{pj} is a covariant tensor of order two. Show that A(p,q) is a mixed tensor.

§ 1.10 Conjugate Symmetric tensor :

Consider a symmetric covariant tensor A_{ij} of order (0,2) such that $|A_{ij}| \neq 0$.

Let us define

1.19) $A^{ij} = \frac{\text{cofactoro } of\ A^{ij} in |A_{ij}|}{|A_{ij}|}$

Then from the theory of determinants

$A_{ij}A^{ik} = 1, \quad j = k$
$\quad\quad\ = 0, \quad j \neq k$

i.e. we may write

1.20) $A_{ij}A^{ik} = \delta_i^k$ by 1.4) of § 1.3

Exercises

1. Show that A^{ij}, defined in 1.19), are the components of a symmetric contravariant tensor of order (2,0).

2. If $A_{ij} = 0$ for $i \neq j$
$\neq 0$, for $i = j$

then show that $A^{ij} = 0$, for $i \neq j$, Further,

$$A^{ii} = \frac{1}{A_{ii}}, \text{(no summation)}$$

Note **:** The tensor A^{ij} defined in 1.19) is also known as the conjugate symmetric tensor of A_{ij}.

§ 1.11 Curvilinear Co-ordinates :

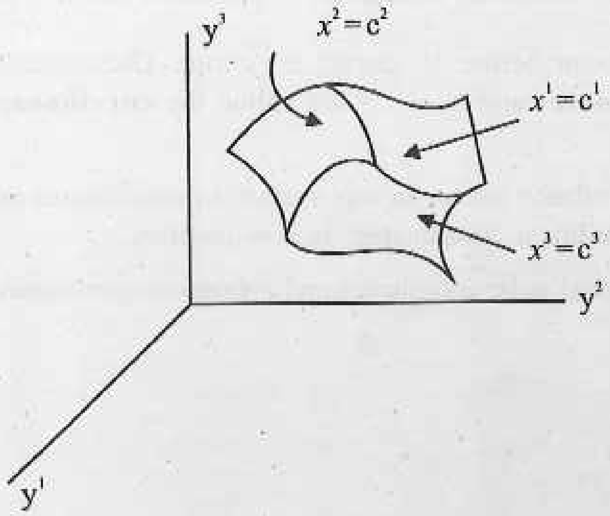
In the beginning of this chapter, we have considered an equation of type 1.1).

Let the co-ordinates x^1, x^2, x^3 be related to the rectangular cartesian co-ordinates y^1, y^2, y^3 by

$$y^i = \phi^i(x^1, x^2, x^3), \quad i = 1, 2, 3$$

Let $x^1 = c^1$, where c^1, is a constant and let x^2, x^3 be allowed to vary. Then

$$y^i = \phi^i(c^1, x^1, x^2),$$



From above it follows that the point $P(y^1, y^2, y^3)$ will lie on a surface which will be denoted by

$x^1 = c^1$

Similar situation will arise when we will consider

$x^2 = c^2$, c^2 being constant

and $x^3 = c^3$, c^3 being constant separately.

Let now $x^1 = c^1$, $x^2 = c^2$, then.

$y^i = \phi^i(c^1, c^2, x^3)$,

$$\text{i.e} \begin{cases} y^1 = \phi^1(c^1, c^2, x^3) \\ y^2 = \phi^2(c^1, c^2, x^3) \\ y^3 = \phi^3(c^1, c^2, x^3) \end{cases}$$

Which shows that the functions ϕ^i are functions of a single variable and hence we say that the point $P(y^1, y^2, y^3)$ lie on a curve. Such a curve, denoted by

$x^i = c^i$, c^i being constant, except for i=3 is called a x^3 – *curve*.

Similarly, we can define x^1 curve, x^2 curve. These curves are also called the **co-ordinate curves** and x^1, x^2, x^3 are called the **curvilinear Co-ordinates** of the point P.

Note. The coordinate curves in this system of coordinates are the curved lines and the name "curvilinear coordinates" is now justified.

Example. Spherical polar coordinates and cylindrical coordinates are the curvilinear coordinates.

UNIT : 2 ❑ RIEMANNIAN SPACE

§ 2.1 The Metric Tensor :

In the three dimensional Euclidean space with Cartesian coordinate system, the distance between tow neighbouring points, say $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ and $(\bar{x}^1 + d\bar{x}^1, \bar{x}^2 + d\bar{x}^2, \bar{x}^3 + d\bar{x}^3)$ is given by

$$ds^2 = (d\bar{x}^1)^2 + (d\bar{x}^2)^2 + (d\bar{x}^3)^2$$

$$= d\bar{x}^i . d\bar{x}^i \quad \text{, say} \quad , i = 1, 2, 3$$

Such ds is called the **lime element** with respect ot orthogonal cartesian coordinatie system.

Note that the line element ds is an invariant, since the distance between the two neighbouring points is independent of the coordinate system.

This idea of distance was extended by Riemann, a German Methematician, to a space of *n*-dimension. In this case we get

2.1) $ds^2 = g_{mn} dx^m dx^n$ where

2.2) $g_{mn} = \dfrac{\partial \bar{x}^i}{\partial x^m} . \dfrac{\partial \bar{x}^i}{\partial x^n}$ are (functions of (x^i) such that

2.3) $g = |g_{mn}| \neq 0$

The quadratic differemtial form $g_{mn}\, dx^m\, dx^n$, which expresses the distance between two neighbouring points, is called a metric or a **Riemannian metric** after Riemann, and g_{mn}*'s* are called the **metric tensor** or the **fundamental metric tensor**. An *n*-dimensional space characterised by such a metric is called a **Riemannian space.**

This is denoted by V^n. Riemannian geometry is that geometry which is based on Riemannian metric.

Theorem 2.1 : The fundamental tensor is a symmetric covariant tensor of order (0, 2).

Proof : Since ds^2 is an invariant and dx^i are the components of an arbitrary contravariant vector, therefore by quotient law we can at one say that

g_{ij} are the components of a covariant tensor of type (o.2).

Clearly,

$$g_{mn} = g_{nm} \text{ from } 2.2$$

Thus we claim that the fundamental tensor is a symmetric covariant tensor of order (0, 2).

This completes the proof.

Let us define

2.5) $$g_{ij} = \frac{cofactor\ of\ g_{ij}\ in\ g}{g}$$

so that g^{ij} and g_{ij} are conjugate or reciprocal tensors i.e.

2.6) $$g_{ij} \quad g_{kj} = \delta^i_k$$

In the same way, as done in the previous chapter, it can be shown that such tensor g^{ij} are the components of a symmetric contravariant tensor of order (2, 0). It is sometimes called the **contravariant fundamental tensor.**

Exerises

1. If g_{ij} denotes the fundamental metric tensor and g^{ij} denotes its reciprocal tensor, then prove that

$$g^{ij} g_{kj} = \delta^i_k$$

2. Prove that g^{ij} is a symmetric contravariant tensor of order (2, 0)
3. Find the metric of an Euclidean space referred to

   a) **cylindrical** and b) **spherical co-ordinates.**

Answer : (a) $ds^2 = \left(d\bar{x}^1\right)^2 + \left(\bar{x}^1\right)^2\left(d\bar{x}^2\right)^2 + \left(\bar{x}^1\right)^2 \sin^2 \bar{x}^2 \left(d\bar{x}^3\right)^2$

(b) $ds^2 = \left(d\bar{x}^1\right)^2 + \left(\bar{x}^1\right)^2\left(d\bar{x}^2\right)^2 + \left(\bar{x}^1\right)^2 \sin^2 \bar{x}^2 \left(d\bar{x}^2\right)^2$

4. Determine the metric tensor and the conjugate metric tensor in a) cylindrical and b) spherical co-ordinates.

5. If the metric is given by

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6dx^1dx^2 + 4dx^2dx^3$$

evaluate g and g^{ij}

Answer : $g = 4, g^{11} = -2, g^{22} = 5,$

$g^{33} = \frac{3}{2}, g^{12} = 3, g^{23} = -\frac{5}{2},$

$g^{13} = -\frac{3}{2}.$

6. Show that

$(g_{hj}\, g_{ik} - g_{hk}\, g_{ih})\, g^{hj} = (n-1)\, g_{ik}$

7. For a V^2 in which

$g_{11} = E, \quad g_{12} = F, \qquad g_{22} = G,$ prove that

$g = EG - F^2, \quad g^{11} = \frac{G}{g}, \quad g^{12} = \frac{F}{g}, \quad g^{22} = \frac{E}{g},$

Solution : 3(a) For cylindrical co-ordinates, it is known that

$y^1 = x^1 \cos x^2, \quad y^2 = x^1 \sin x^2, \qquad y^3 = x^3$

Thus,

$$dy^1 = dx^1 \cos x^2 - x^1 \sin x^2\, dx^2$$

$$dy^2 = dx^1 \sin x^2 + x^1 \cos x^2 dx^2$$

$$dy^3 = dx^3$$

Hence from

$$ds^2 = dx^i dx^i, \quad i = 1, 2, 3$$

one must have

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (dx^3)^2$$

Which is the metric of an Euclidean space referred to cylindrical co-ordinates.

§ 2.2 Associated Vectors, Magnitude of Vectors, Angles :

Let $A^{m_1 \ldots m_p}_{n_1 \ldots n_q}$ be a given tensor of order (p, q). All tensors obtained from the given one, by performing inner multiplication with the metric tensor g_{ij} and its conjugate g^{ij} are called **associated tensors** of the given tensor.

To explain it, consider a contravariant vector A^j and a covariant vector B_k in (x^i) system. Define A_i and B^i as follows :

2.7) $A_i = g_{ij} A^j$ and

2.8) $B^i = g^{ik} B_k$

Then the associate to a given contravariant vector A^j is formed by **lowering** its index by the fundamental metric tensor g_{ij} and the associate to a given covariant vector B_k is formed by **raising** its index by the conjugate metric tensor g^{ij} . Also

2.9) $g^{ik} A_i = A^k$

Conequently, the associate to A_i is A^k. Hence if A_i is the associate to A^i, then, A^i is the associate to A_i : Thus, A_i and A^i and mutually **associate.**

Let A and B denote the magnitude or length of the contravariant vector A^i and the covariant vector B_k respectively. We define.

2.10) $A^2 = g_{ij} A^i A^j$ and

2.11) $B^2 = g^{ik} B_i B_k$

A vector with unity as magnitude is called a **unit vector.** In this case

2.12) $g_{ij} A^i A^j = 1 = g^{ik} B_i B_k$

A vector, whose magnitude is zero is called a **null vector.** In that case

2.13) $g_{ij} A^i A^j = 0 = g^{ik} B_i B_k$

The angle θ between two non-null vectors A^i and B^i is defined as follows

2.14) $$Cos\theta = \frac{g_{ij} A^i B^j}{\sqrt{g_{ij} A^i A^j} \sqrt{g_{ij} B^i B^j}}$$

Exercises

1. Show that the magnitudes of two associated vectors are same
2. Show that $\frac{dx^i}{ds}$ is a unit contravariant vector.
3. Prove that the length of a vector is invariant.
4. Prove that the necessary and sufficient condition for two vectors A^i, B^i to be orthogonal is that $g_{ij} A^i B^j = 0$
5. If θ is the angle between two non-null vectors A^i and B^i show that

$$\sin^2\theta = \frac{\left(g_{ij}g_{pq} - g_{ip}g_{jq}\right)A^i A^j B^p B^q}{\left(g_{ij}A^i A^j\right)\left(g_{pq}B^p B^q\right)}$$

6. If p^i and q^i are orthogonal unit vectors, show that

$$(g_{hj}\, g_{ik} - g_{hk}\, g_{ij})p^h p^j q^i q^k = 1$$

§ 2.3 Christoffel Symbols :

We now consider two expressions due to christoffel involving the derivatives of the components of the fundamental tensors g_{ij} and g^{ij}. The christoffel symbols of the 1st kindand the 2nd kind are denoted respectively by [ij, k] and $\left\{ {k \atop ij} \right\}$ and are defined as

2.15) $[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$ and

2.16) $\left\{ {k \atop ij} \right\} = g^{km}[ij,m]$

Exercises

1. Show that the christoffel symbols defined in 2.15) and 2.16) are symmetric in i, j.

2. Prove that the necessary and sufficient condition that all the christoffel symbols vanish at a point is that g_{ij}'s are constant.

3. Deduce that

a) $[ij,k] + [kj,i] = \frac{\partial g_{ik}}{\partial x^j}$

b) $[ij,k] + [ij,k] = \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}$

c) $\frac{\partial g_{ik}}{\partial x^j} = -g^{pk}\left\{ {i \atop pj} \right\} - g^{im}\left\{ {k \atop mj} \right\}$

d) $\left\{ {i \atop ij} \right\} = \frac{\partial}{\partial x^j}\left(\log\sqrt{g}\right), \quad g \neq 0$

4. Evaluate the christoffel symbols of both kinds for spaces where $g_{ij} = 0$, *if* $i \neq j$

5. Calculate the christoffel symbols of the 2nd kind in (a) rectangular b) cylindrical and c) Spherical co-ordinates.

6. Evaluate [11. 2], [12, 2] in cylindrical co-ordinates.

Solution : 4. Since $g_{ij} = 0$, $i \neq j$, we have the following cases

Case a) : $i = j = k$

$$[ii,i] = \frac{1}{2}\frac{\partial g_{ii}}{\partial x_i}$$

Case b) : $i = j$, $i \neq k$, $j \neq k$

$$[ii,k] = -\frac{1}{2}\frac{\partial g_{ii}}{\partial x^k}$$

Case c) : $i = k$, $i \neq j$, $k \neq j$

$$[ij,i] = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^j}$$

Also

$$\left\{ {k \atop ij} \right\} = g^{km}[ij,m] = g^{kk}[ij,k], \quad as \quad g^{km} = 0, k \neq m$$

$= \frac{1}{g_{kk}}[ij,k]$ as the co-ordinate system is orthogonal.

Case a) : $\left\{ {i \atop ii} \right\} = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{ii}}{\partial x^i}$

Case b) : $\left\{ {k \atop ii} \right\} = \frac{1}{2g_{kk}} \cdot \frac{\partial g_{ii}}{\partial x^k}$

Case c) : $\left\{ {i \atop ij} \right\} = \frac{1}{2g_{kk}} \cdot \frac{\partial g_{ii}}{\partial x^j}$

5. b) The only non-vanishing christoffel symbols are the following

$$\left\{ {1 \atop 22} \right\} = -x^1 \quad and \quad \left\{ {1 \atop 12} \right\} = \left\{ {1 \atop 21} \right\} = \frac{1}{x^1}$$

we note that

$$\overline{g}_{ln} = g_{ij} \frac{\partial x^i}{\partial \overline{x}^l} \cdot \frac{\partial x^j}{\partial \overline{x}^n}$$

On differentiating with respect to $\overline{x}^m$, we find

$$\frac{\partial \overline{g}_{ln}}{\partial \overline{x}^m} = \frac{\partial g_{ij}}{\partial x^k} \cdot \frac{\partial x^k}{\partial \overline{x}^m} \cdot \frac{\partial x^i}{\partial \overline{x}^l} \cdot \frac{\partial x^j}{\partial \overline{x}^n} + g_{ij} \frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^m} \cdot \frac{\partial x^j}{dx^{-n}} + g_{ij} \frac{\partial x^i}{\partial \overline{x}^l} \cdot \frac{\partial^2 x^i}{\partial \overline{x}^n \partial \overline{x}^m}$$

Now from the definition

$$\overline{[lm.n]} = \frac{1}{2}\left(\frac{\partial \overline{g}_{ln}}{\partial \overline{x}^m} + \frac{\partial \overline{g}_{mn}}{\partial \overline{x}^l} - \frac{\partial \overline{g}_{lm}}{\partial \overline{x}^n} \right)$$

using the above result, one gets after a few steps

$$2.17) \quad \overline{[lm,n]} = [ik,j] \frac{\partial x^i}{\partial \overline{x}^l} \cdot \frac{\partial x^j}{\partial \overline{x}^n} \cdot \frac{\partial x^k}{\partial \overline{x}^m} + g_{ij} \frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^m} \cdot \frac{\partial x_j}{\partial \overline{x}^n}$$

The existence of the second term on the right hand side of 2.17) shows that the christoffel symbols of the 1st kind is not a tensor. Again

$$\overline{g}^{rs} = g^{pq} \frac{\partial \overline{x}^r}{\partial x^p} \cdot \frac{\partial \overline{x}^s}{\partial x^q}$$

Inner nultiplication of both sides of 2.17) by the corresponding sides of the above equation, we get

$$2.18)\quad \overline{\left\{ {s \atop lm} \right\}} = \left\{ {q \atop ik} \right\} \frac{\partial x^i}{\partial \overline{x}^l} \cdot \frac{\partial x^k}{\partial \overline{x}^m} \cdot \frac{\partial \overline{x}^s}{\partial x^q} + \frac{\partial^2 x^q}{\partial \overline{x}^l\, \partial \overline{x}^m} \cdot \frac{\partial \overline{x}^s}{\partial x^q}$$

The existence of the second term on the right hand side of

2.18) shows that the christoffel symbols of the 2nd kind is not a tensor.

Inner mulitiplication of 2.18) with $\frac{\partial x^r}{\partial \overline{x}^s}$ one gets

$$2.19)\quad \frac{\partial^2 x^r}{\partial \overline{x}^l\, \partial \overline{x}^m} = \overline{\left\{ {s \atop lm} \right\}} \frac{\partial x^r}{\partial \overline{x}^s} - \left\{ {r \atop ik} \right\} \frac{\partial x^i}{\partial \overline{x}^l} \cdot \frac{\partial x^k}{\partial \overline{x}^m}.$$

§ 2.4 Covariant Differentiation of Vectors and tensors :

From the transformation law of a contravariant vector we know that

$$A^k = \overline{A}^i \frac{\partial x^k}{\partial \overline{x}^i}$$

Differentiating with respect to x^j we get

$$\frac{\partial A^k}{\partial x^j} = \frac{\partial \overline{A}^i}{\partial \overline{x}^n} \cdot \frac{\partial \overline{x}^n}{\partial x^j} \frac{\partial x^k}{\partial \overline{x}^i} + \overline{A}^i \frac{\partial^2 x^k}{\partial \overline{x}^i\, \partial \overline{x}^n} \cdot \frac{\partial \overline{x}^n}{\partial x^j}$$

using 2.19) we find from above, after a few steps—

$$= \frac{\partial \overline{A}^i}{\partial \overline{x}^n} \cdot \frac{\partial \overline{x}^n}{\partial x^j} \frac{\partial x^k}{\partial \overline{x}^i} + \overline{\left\{ {s \atop in} \right\}} \overline{A}^i \frac{\partial x^k}{\partial \overline{x}^s} \cdot \frac{\partial \overline{x}^n}{\partial x^j} - \left\{ {k \atop uj} \right\} A^u$$

$$\text{or}\quad \frac{\partial A^k}{\partial x^j} + \left\{ {k \atop uj} \right\} A^u = \left(\frac{\partial \overline{A}^i}{\partial \overline{x}^n} + \left\{ {s \atop in} \right\} \overline{A}^i \right) \frac{\partial x^k}{\partial \overline{x}^s} \cdot \frac{\partial \overline{x}^n}{\partial x^j}$$

Writing

$$2.20)\quad A^k_{,j} = \frac{\partial A^k}{\partial x^j} + A^u \left\{ {k \atop uj} \right\}$$

one gets from above

$$A^k_{,j} = \overline{A}^s_{,n} \frac{\partial x^k}{\partial \overline{x}^s} \cdot \frac{\partial \overline{x}^n}{\partial x^j}$$

Which is the law of transformation of a mixed tensor of order (1, 1). Such a tensor $A^k_{,j}$ is defined to be the **covariant derivative of the contravariant vector A^k.**

Exercises

1. If A^i is a contravariant vector, prove that

$$A^i_{,i} = \frac{1}{\sqrt{g}} \cdot \frac{\partial}{\partial x^i}\left(\sqrt{g} A^i\right), \quad g = |g_{ij}|$$

Such $A^i_{,i}$ is called the **divergence** of the vector A^i and is denoted by div A^i. Thus

2.21) div $A^i = A^i_{,i}$

2. Show that the the covariant derivative of a covariant vector with components A_k is given by

$$\overline{A}_{k,j} = A_{s,n} \frac{\partial x^s}{\partial \overline{x}^k} \cdot \frac{\partial x^n}{\partial \overline{x}^j}$$

where

2.22) $A_{k,j} = \dfrac{\partial A_k}{\partial x^j} - A_m \left\{ {m \atop kj} \right\}$

3. Show that

$$A_{i,j} - A_{j,i} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

The tensor $A_{i,j} - A_{j,i}$ of order (0, 2) is defined as the curl of the vector A_i and is denoted by A_i. Thus

2.23) curl $A_i = A_{i,j} - A_{j,i}$

4. Show that the covariant derivative of a mixed tensor with components A^i_j is given by

$$\bar{A}^i_{j,k} = A^p_{q,r} \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial x^q}{\partial \bar{x}^j} \cdot \frac{\partial x^r}{\partial \bar{x}^k} \quad \text{, where}$$

$$2.24) \quad A^p_{q,r} = \frac{\partial A^p_q}{\partial x^r} + A^s_q \left\{ {p \atop sr} \right\} - A^p_s \left\{ {s \atop qr} \right\}$$

5. Prove that the covariant derivative of the tensors g_{ij}, g^{ij} and δ^i_j vanish identically.

Note : The above result is also known as RICCI's Theorem.

6. Show that $\frac{\partial}{\partial x^k}(g_{ij} A^i B^j) = A_{i,k} B^i + A^i B_{i,k}$

7. If $A_i = g_{ij}, A^j$, show that $A_{i,k} = g_{i\alpha}, A^\alpha_{,k}$,

In general

$$2.25). \quad A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_q, n} = \frac{\partial A^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q}}{\partial x^n} + A^{\alpha i_2 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q} \left\{ {i_1 \atop \alpha n} \right\} + \ldots\ldots A^{i_1 i_2 \ldots\ldots i_{p-1} \alpha}_{j_1 \ldots\ldots j_q} \left\{ {i_p \atop \alpha n} \right\}$$

$$- A^{i_1 \ldots\ldots i_p}_{\beta j_2 \ldots\ldots j_q} \left\{ {\beta \atop j_1 n} \right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - A^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_{q-1} \beta} \left\{ {\beta \atop j_q n} \right\}$$

§ 2.5 Riemann -christoffel tensor and its properties :

A further covariant differentiation of

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - A_m \left\{ {m \atop ij} \right\}$$

yields

$$(A_{i,j})_{,k} = A_{i,jk}$$

and is knows as the second covariant derivative of the given covariant vector A_i, Thus

$$(A_{i,j}),k = \frac{\partial A_{i,j}}{\partial x^k} - A_{p,j}\left\{ {p \atop ik} \right\} - A_{i,p}\left\{ {p \atop jk} \right\}$$

using 2.22) one gets from above

$$A_{i,jk} = \frac{\partial^2 A_i}{\partial x^k \partial x^j} - \frac{\partial A_m}{\partial x^k}\left\{ {m \atop ij} \right\} - A_m \frac{\partial}{\partial x^k}\left\{ {m \atop ij} \right\} - \frac{\partial A_p}{\partial x^j}\left\{ {p \atop ik} \right\} + A_s\left\{ {s \atop pj} \right\}\left\{ {p \atop ik} \right\}$$

$$- \frac{\partial A_i}{\partial x^p}\left\{ {p \atop jk} \right\} + A_s\left\{ {s \atop ip} \right\}\left\{ {p \atop jk} \right\}$$

It can be shown that

2.26) $A_{i,jk} - A_{i,kj} = A_m R^m_{ijk}$ where

2.27) $R^m_{ijk} = \frac{\partial}{\partial x^j}\left\{ {m \atop ik} \right\} - \frac{\partial}{\partial x^k}\left\{ {m \atop ij} \right\} + \left\{ {p \atop ik} \right\}\left\{ {m \atop pj} \right\} - \left\{ {p \atop ij} \right\}\left\{ {m \atop pk} \right\}$

is called the Reemann-christoffel tensor of the second kind or the **curvature tensor** of the Riemannian space. Applying quotient law, it is evident from 2.27) that R^m_{ijk} is a mixed tensor of order (1, 3)

The associated tensor

2.28) $R_{hijk} = g_{hm} R^m_{ijk}$

is called the Riemann-christoffel tensor of the **1st kind.**

From 2.27) we find that

$$R^m_{ijk} = \frac{\partial}{\partial x^j}\left\{ {m \atop im} \right\} - \frac{\partial}{\partial x^m}\left\{ {m \atop ij} \right\} + \left\{ {p \atop im} \right\}\left\{ {m \atop pj} \right\} - \left\{ {p \atop ij} \right\}\left\{ {m \atop pm} \right\}$$

using Exercise 3(d) of § 2.3, we find

2.29) $R^m_{ijk} = \frac{\partial^2}{\partial x^j \partial x^i}(\log\sqrt{g}) - \frac{\partial}{\partial x^m}\left\{ {m \atop ij} \right\} + \left\{ {p \atop im} \right\}\left\{ {m \atop pj} \right\} - \left\{ {p \atop ij} \right\}\frac{\partial}{\partial x^p}(\log\sqrt{g})$

such a contraction defined by

2.30) $R_{ij} = R^{m}_{ijm}$

is called a **Ricci tensor.**

The **scalar curvature** R is defined by

2.31) $R = g^{ij} R_{ij}$

A space for which

2.32) $R_{ij} = \lambda g_{ij}$, λ being invariant at all points is called an **Einstein space**.

Exercises

1. Show that

   i) $R^{m}_{ijk} = 0$

   ii) $R^{m}_{ijk} = R^{m}_{ikj}$

   iii) $R^{m}_{ijk} + R^{m}_{jki} + R^{m}_{kij} = 0$

2. Show that

   (i) $R_{iijk} = - R_{iihjk}$ (ii) $R_{hijk} = - R_{hikj}$ (iii) $R_{hijk} = R_{jkhi}$ (iv) $R_{hijk} + R_{hjki} + R_{hkij} = 0$

3. Prove that the Ricci tensor is symmetric.

4. Show that for an Einstein space of dimension **n ≥ 2**

   $$R_{ij} = \frac{R}{n} g_{ij}$$

5. Prove that the scalar curvature of an Einstein space is constant provided $n > 2$.

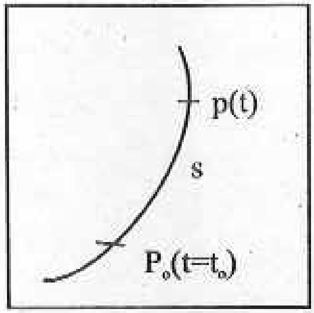
UNIT : 3 ◻ CURVES IN SPACE

§ 3.1 Intrinsic derivative :

In the previous chapter, we have introduced covariant differentiation in a Riemannian space. Such a concept is regarded as a generalisation of partial differentiation in Euclidean space with rectangular cartesion co-ordinates, as in this case,

$$A^i_{,m} = \frac{\partial A^i}{\partial x^k}$$

for $\left\{ {}^{\,i}_{jk} \right\} = 0$ (see Exericse 5 (a) of § 2.3)



We are now going to introduce another kind of differentation which may be regarded as a generalisation of ordinary differentiation in Euclidean space with rectangular cartesian co-ordinates.

A space curve C is the totality of points whose co-ordinates x^i satisfy equations of the form,

3.1) $C:\ x^i = \phi^i\ (t)$

where ϕ^i are functions of a single parameter t.

The **length** s of the curve C from a point P_o on C to the variable point P one C, corresponding to the value t_o and t respectively, is defined as follows.

$$3.2) \qquad s = \int_{t_0}^{t} \sqrt{g_{ij} \frac{dx^i}{dt} \cdot \frac{dx^j}{dt}}\ dt$$

Consider a tensor of order (p, q) with components $A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_q}$ say, such that they are the functions of a single parameter t. Then the intrinsic derivative of such a tensor with respect to t is denoted by $\frac{\delta}{\delta t} A^{i_1 \ldots\ldots i_p}_{j_1 \ldots\ldots j_q}$ and is defined as follows :

3.3) $$\frac{\delta}{\delta t} A^{i_1 \ldots i_p}_{j_1 \ldots j_q} = A^{i_1 \ldots i_p}_{j_1 \ldots j_q}{}_{,k} \frac{dx^k}{dt}$$

where comma (,) denotes covariant differentiation.

Hence the intrinsic derivative of a tensor of order (1, 0) and (0, 1) along the curve are defined as respectively as

3.4) $$\frac{\delta A^i}{\delta t} = A^i_{,k} \frac{dx^k}{dt} = \left(\frac{\partial A^i}{\partial x^k} + A^m \left\{ {i \atop ml} \right\} \right) \frac{dx^k}{dt} = \frac{dA^i}{dx^k} + A^m \left\{ {i \atop ml} \right\} \frac{dx^k}{dt}$$

3.5) $$\frac{\delta A^i}{\delta t} = \frac{dA_i}{dt} - A_m \left\{ {m \atop ik} \right\} \frac{dx^k}{dt}$$

Exercises

1. Show that the intrinsic derivative of an invariant coincides with its total derivative.
2. Prove that the intrinsic derivative of the fundamental tensors and the kronecker delta are xero.
3. Show that

   (i) $\frac{d}{dt}(g_{ij} A^i A^j) = 2 g_{ij} A^i \frac{dA^j}{dt}$

   (ii) $\frac{d}{dt}(g_{ij} A^i B^j) = g_{ij} \frac{\delta A^i}{\delta t} B^j + g_{ij} A^i \frac{\delta B^j}{\delta t}$

4. If the intrinsic derivative of a vector A along a curve C vanishes at all points of C, show that the magnitude of A is constant along C.

§ 3.2 Serret -Frenet formulii :

Let a curve C be given by the equation

$C : \ x^i = x^i \ (s)$

where the parameter s measures the are distance along curve. The square of the length of the elements of are of C is given by

$ds^2 = g_{ij} dx^i dx^j$

from where one gets

3.6) $\quad 1 = g_{ij} \dfrac{dx^i}{ds} \cdot \dfrac{dx^j}{ds}$

showing that $\dfrac{dx^i}{ds}$ is a unit vector.

Let P be a given point with co-ordinates (x^i) and Q be a neighbouring point with co-ordinates ($x^i + dx^i$) on C, corresponding to an incrementi ds in the arc.

Then the vector $\underset{Q \to P}{Lt} \dfrac{\vec{P}Q}{ds}$ is called the **tangent vector** and we shall denote it by λ^i. Thus

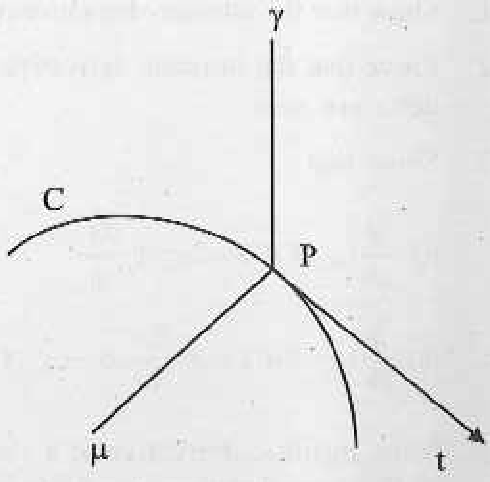
3.7) $\quad \lambda^i = \dfrac{dx^i}{ds}$

and hence from 3.6) we find that

3.8) $\quad I = g_{ij} \lambda^i \lambda^j$

Taking the intrinsic derivative of 3.8) with respect to s, we find

$g_{ij} \lambda^i \dfrac{\delta \lambda^j}{\delta s} = 0$



Thus either $\dfrac{\delta \lambda^j}{\delta s}$ vanishes or is orthogonal to λ^i. Any vector which is orthogonal to the tangent vector at some point of a curve is said to be a **normal vector** of that curve at that point. Hence the condition that a vector μ^i be normal to C at P is

3.9) $g_{ij} \lambda^i \mu^j = 0$

Thus if $\frac{\delta \lambda^j}{\delta s}$ does not vanish, we denote the unit

vector codirectional with $\frac{\delta \lambda^j}{\delta s}$ by μ^i and write it as

3.10) $\mu^j = \frac{1}{x} \frac{\delta \lambda^j}{\delta s}$

where $x > 0$ is chosen to make μ^j a unit vector. The vector determined by 3.10) is called the **principal normal vector** to the curve C at P and x is called the **curvature** of C at the given point.

Since μ is a unit vector

3.11) $g_{ij} \mu^i \mu^j = 1$

Taking its intrinsic derivative we find

3.12) $g_{ij} \mu^i \frac{\delta \mu^j}{\delta s} = 0$

We take the intrinsic derivative of 3.9 and on using 3.10), 3.11) we get

$$g_{ij} \lambda^i \frac{\delta \mu^j}{\delta s} = -x$$

Using 3.8) we find

$$g_{ij} \lambda^i \left(\frac{\delta \mu^j}{\delta s} + x^{ji} \right) = 0, \text{ since } g_{ij} \lambda^i \lambda^j = 1.$$

Also from 3.9) and 3.12) we find $g^{ij} \quad \mu^i \quad \left(\frac{\delta \mu^j}{\delta s} + x \lambda^j \right) = 0$

Thus $\frac{\delta \mu^j}{\delta s} + x \quad \lambda^j$ is orthogonal to λ^i and μ^i simultaneously Hence we define a vector γ^i by

3.13) $\nu^i = \frac{1}{\tau}\left(\frac{\delta\mu^i}{\delta s} + x\lambda^i\right)$

where τ is chosen to make ν a unit vector. The sign of τ is not always positive but we agree to choose the sign of τ in such a way that (λ, μ, ν) form a right handed system i.e.

3.14) $\epsilon_{ijk}\, \lambda^i\, \mu^j\, \nu^k = +1$ where

3.15) $\epsilon_{ijk} = \sqrt{g}\; e_{ijk}$; $\epsilon^{ijk} = \frac{1}{\sqrt{g}} e^{ijk}$ and

3.16) $$\begin{cases} e_{123}(e^{123}) = e_{231} = e_{312} = +1 \\ e_{132} = e_{213} = e_{321} = -1 \\ e_{111} = e_{222} = e_{333} = 0 \end{cases}$$

The vector ν is called the **binormal** of C at P and τ is called the **torsion** of C.

Exercises

1. Show that

i) $x = \left(g_{mn} \frac{\delta\lambda^m}{\delta s} \cdot \frac{\delta\lambda^n}{\delta s} \right)^{1/2}$

ii) $\tau = \epsilon_{rst}\, \lambda^r \mu^s \frac{\delta\mu^t}{\delta s}$

iii) $\gamma = \epsilon^{kij}\, \lambda_i\, \mu_j$ where λ_i, μ_j are the associated vectors of λ^i and μ^j

iv). $\mu_i = \frac{1}{x} \cdot \frac{\delta\lambda_i}{\delta s}$, $\nu_i = \frac{1}{\tau}\left(\frac{\delta\mu_i}{\delta s} + x\lambda_i\right)$

v) $x\nu_i = \epsilon_{ijk}\, \lambda^j \frac{\delta\lambda^k}{\delta s}$

From Exercise 1 (iii) above, we see that $\frac{\delta \nu^k}{\delta s} = -\in^{ikj} \lambda_i \nu_j \tau$

3.17) $\frac{\delta \nu^k}{\delta s} = -\tau \mu^k$

Thus the following three relations

3.18) $$\begin{cases} \frac{\delta \lambda^i}{\delta s} = x\mu^2 \\ \frac{\delta \mu^i}{\delta s} = \tau \nu^i - x\lambda^i \\ \frac{\delta \nu^i}{\delta s} = -\tau \mu^i \end{cases}$$

are know as the **serret - Frenet** formulii for a space curve.

Exercises

1. Find the curvature and torsion at any point of the

   i) Curve $C : x^1 = a, \quad x^2 = t, \quad x^3 = 0$

   ii) Curve $C : x^1 = a, \quad x^2 = t, \quad x^3 = ct$

where

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (dx^3)^2$$

[Answer : (ii) $x = \frac{a}{a^2 + c^2}, \quad \tau = \frac{c}{a^2 + c^2}$

2. Show that the tangent vector derived in Exercise 1 (ii) above makes a constant angle with the vector (0, 0, 1).

3. Show that a space curve is a straight line if and only if its curvature is zero at all points of it.

4. Prove that a space curve is a plane curve if and only if its torsion is xero at all points.

Solution : 1 (i) In cylindrical co-ordinates, it is known that the only non-zero christoffel symbols are $\left\{ {1 \atop 22} \right\} = -x^1$ and $\left\{ {1 \atop 22} \right\} = \left\{ {2 \atop 21} \right\} = \frac{1}{x^1}$.

Now $\lambda^m = \frac{dx^m}{ds}$ Hence for the given curve C, the components of $\lambda\left(0, \frac{dt}{ds}, 0\right)$. As are $g_{mn}\ \lambda^m \lambda^n = 1$, we must have,

$$\frac{dt}{ds} = \frac{1}{a}$$

$$\therefore\ \lambda = \left(0, \frac{1}{a}, 0\right).$$

From serret-Frenet formula $\frac{\delta\lambda^r}{\delta s} = x\mu^r$, $r = 1,2,3$ we see that

$$x\mu^1 = \frac{\delta\lambda^1}{\delta s} = \frac{d\lambda^1}{ds} + \left\{ {1 \atop mn} \right\} \lambda^m \frac{dx^n}{ds} = \frac{d\lambda^1}{ds} + \left\{ {1 \atop 22} \right\} \lambda^2 \frac{dx^2}{ds} = -\frac{1}{a}$$

$$x\mu^2 = 0 \qquad , \qquad x\mu^3 = 0$$

Also,

$$g_{mn}\ \lambda^m \lambda^n = 1$$

$$\therefore\ 1.\frac{1}{a^2} = x^2$$

$$x = \frac{1}{a}, \qquad as\ k > 0$$

$$\therefore \mu = (-1,\ 0,\ 0)$$

Also from

$$\tau\nu^r - k\lambda^r = \frac{\delta\mu^r}{\delta s},\ r = 1,\ 2,\ 3$$

it can be easily shown that

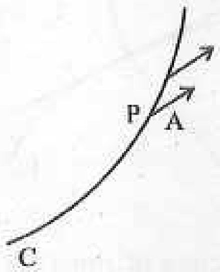
$$\tau\nu^1 - x\lambda^1 = 0, \quad \tau\nu^2 - x\lambda^2 = -\frac{1}{a^2}, \tau\nu^3 - x\lambda^3 = 0$$

After a few steps, it can be calculated that

$\tau = 0, \nu = (0, 0, 1)$

§ 3.3 Parallel Vector Field :

Consider a curve C given by 3.1) and a vector A localised at some point P of C whose components A^i are functions of t. If we construct at every point of C, a vector equal to a in magnitude and parallel to it in direction, we obtaiñ what is known as a **parallel field of vectors along C.** We say that the vector A suffers a parallel displacement along C if



$$3.19) \quad \frac{\delta A^i}{\delta t} = 0$$

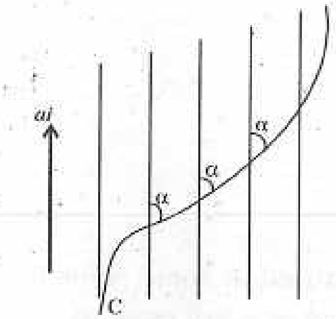
Exercises

1. If A^i and B^i are two vectors of constant magnitudes and undergo parallel displacements along a given curve, then show that they are inclined at a constant angle.

2. If $(a\lambda^r + b\mu^r + c\nu^r)$ forms a parallel vector field along C, prove that

$$\frac{da}{ds} - xb = 0, \quad \frac{db}{ds} + xa - \tau c = 0, \quad \frac{dc}{ds} + \tau b = 0, \ a, b, c \text{ are scalars.}$$

§ 3.4 Helix :

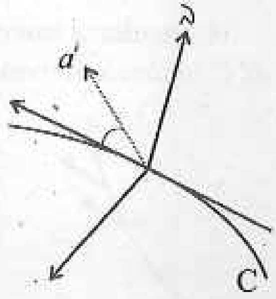


Let

$C: \ x^i = x^i(s)$

be the equation of a curve which makes a constant angle α with a fixed direction a^i, where a^i is a unit vector. Thus

3.20) $\cos\alpha = g_{ij} a^i \lambda^j$



Differentiating 3.20) intrinsically with respect to s we get

3.21) $g_{ij} a^i \mu^j = 0$ as $k \neq 0$, $\quad \dfrac{\delta\lambda^j}{\delta s} = k\mu^j$

Thus a^i is orthogonal to μ^i . But λ^i is also orthogonal μ^i . Hence a^i must lie in the plane determined by λ^i and. μ^i. Consequently the angle between a^i and ν^i will be $90° - \alpha$. Thus

$\cos(90° - \alpha) = g_{ij} a^i \nu^j$

or 3.22) $\sin\alpha = g_{ij} a^i \nu^j$

Again, differentiating intrinsically, we get from 3.21)

$g_{ij} a^i \tau \nu^j - g_{ij} a^i x \lambda^j = 0$ by) 3.18)

or $\tau \sin\alpha - x \cos\alpha = 0$ by 3.20), 3.22)

or $\dfrac{x}{\tau} = \tan\alpha =$ constant

We are now going to define a space curve called helix as follows :

A space curve is called a **helix** if the tangent at every point of it makes a constant angle with a fixed direction.

Exercises

Show from Frenet's formulii, that when $\tau/_x =$ constant and the coordinates are cartesian.

$y^i = c\lambda^i + b^i$.

where C and b^i are constant and also that tangent vector makes a constant angle with a fixed direction.

Solution. In Cartesian coordinate system from Frenet's formulae we get

$$\frac{d\lambda^i}{ds} = k\mu^i, \frac{dv^i}{ds} = -\tau\mu^i$$

Therefore, we get $\frac{d\lambda^i}{dv^i} = -\frac{-k}{\tau} = -a$ (say)

or, $d\lambda^i = -adv^i$ which implies $v^i = c\lambda^i + b^i$,

where $c = -\frac{1}{a}$ and b^i is a constant.

Since $\frac{\tau}{k} =$ constant = a (say), we get

$$\frac{dv^i}{ds} = -\tau\mu^i = -ak\mu^i = -a\frac{d\lambda^i}{ds}$$

So, $\frac{d}{ds}(v^i + a\lambda^i) = 0$ which implies $v^i + a\lambda^i$ is a constant vector. We denote this vector by c^i.

Now $g_{ij}c^i\lambda^j = g_{ij}(v^i + a\lambda^i)\lambda^j = a =$ constant.

This shows that λ^i makes a constant angle with a fixed direction c^i.

This completes the solution.

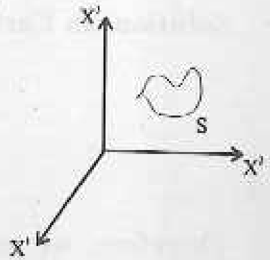
UNIT : 4 ☐ SURFACES IN SPACE

§ 4.1 Curvilinear Co-ordinates on a surface :

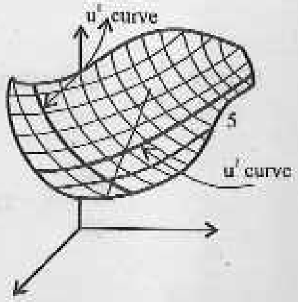
A surface S is defined, in general, due to Gauss, as the set of points whose co-ordinates are functions of two independent parameters. Thus the equation of a surface is of the form

4.1) $s : x^i = x^i (u^1, u^2)$, $i = 1, 2, 3$

where u^1, u^2 are parameters, and (x^i) is the system of rectangular cartesian co-ordinates.



In the remainder of this chapter, we will study the properties of surfaces imbedded in a three dimensional Euclidean Space. It will be shown that certain of these properties can be phrased independently of the space in which the surface is immersed and that they are concerned solely with the structure of the differential quadratic form for the element of arc of a curve drawn on the surface. All such properties of surfaces are termed the intrinsic properties and the geometry based on the study of this differential quadratic form is called the **intrinsic geometry of the surface.**



Any point on a surface is uniquely determined dy u^1, u^2 and we can therefore call these quantities the coordinates of a point on the surface.

Let us examine the geometrical significance of (u^1, u^2). It u^2 is kept constant and u^1 alone varies, then the point depends upon a single parameter and therefore describes a curve. Such curve lies wholly on the surface and we see that when we give u^2 a series of constant values, we obtain a family of curves on the surface. Thus u^1 curve is characterised by the equation

4.2) u^1 curve : u^2 = constant.

Similarly, we have another family of curves, given by u^1 = constant, along which u^2 varies. Each of these is called a u^2 curve and it is characterised by the equation

4.3) u^2 curve : u^1 = constant.

We shall often refer to the u^1 and u^2 curves briefly as the **parametric curves** and we shall call u^1, u^2, a system of curvilinear **or Gaussian coordinates** of the surface.

Note that the parametric representation of a surface is not unique and there are infinitely many curvilinear coordinate system which can be used to locate points on a given surface s.

EXERCISES

1. On the surface of revolution
   $x = u\cos\phi,\ y = u\sin\phi,\ z = f(u)$
   what are the parametric curves?
2. On the right helicoid given by
   $r = (u\cos\phi,\ u\sin\phi,\ c\phi)$
   find the parametric curves for u = constant and ϕ = constants.

Ans. i) U curve : (ucosk, usink, f(u), k = const.
ϕcurve : (ccosϕ, csinϕ, f(c), c = const.
ii) (dcosϕ, dsinϕ, cϕ) and
(ucosk, usink, ck)

§ 4.2 The element of length and the metric tenson :

Let P and Q be two neighbouring points on a surface S with coordinates u^α and $u^\alpha + du^\alpha$ respectively. We shall denote by x^i and x^i+dx^i, the cartesian coordinates of P and Q respectively in space. From (4.1)

4.4) $$dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha$$

If we let ds be the elementary distance between P and Q then

$$ds^2 = \sum_i dx^i dx^i$$

and hence from 4.4) we must have

4.5) $ds^2 = a_{\alpha\beta} du^\alpha du^\beta$, say, where

4.6) $a_{\alpha\beta} = \sum_i \frac{\partial x^i}{\partial u^\alpha} \cdot \frac{\partial x^i}{\partial u^\beta}$

As discussed in § 2.1 we can show in similar way, that, $a_{\alpha\beta}$ is a symmetric covariant tensor of order two. The experession for ds^2 given by 4.5) is the square of the linear element of C lying on the surface S and the right hand member of 4.5) is called the **first fundamental form** of the surface.

If we denote the determinant $|a_{\alpha\beta}|$ by a and define

4.7) $a^{\alpha\beta} = \frac{\text{cofactor of } a_{\alpha\beta} \text{ in } a}{a}$, then

4.8) $a^{\alpha\beta} a_{\beta\gamma} = \delta^\alpha_\gamma$

and it can be proven that $a^{\alpha\beta}$ is a contravariant tensor of order two. Note that

4.9 $\begin{cases} \in_{\alpha\beta} = \sqrt{a}\, e_{\alpha\beta} & e_{11} = e_{22} = 0 \\ \in^{\alpha\beta} = \sqrt{\frac{1}{a}}\, e^{\alpha\beta} & e_{12} = 1, e_{21} = -1 \end{cases}$

Exercises

1. Find ds^2 for the following surfaces :

 i) $x^1 = a \cos u^1$, $x^2 = a \sin u^1$, $x^3 = u^2$

 ii) $x^1 = u^1 \cos u^2$, $x^2 = u^1 \sin u^2$, $x^3 = 0$

 ii) $x^1 = a \cos u^1 \cos u^2$, $x^2 = a \cos u^1 \sin u^2$, $x^3 = a \sin u^1$

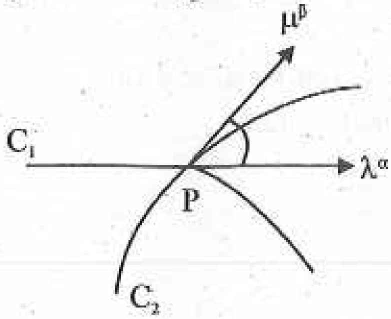
 where x^1 are Orthogonal cartesian co-ordinates.

Ans. i) $ds^2 = a^2 (du^2)^2 + (du^2)^2$

ii) $ds^2 = a^2 (du^1)^2 + (u^1)^2 (du^2)^2$

iii) $ds^2 = a^2 (du^1)^2 + a^2 \cos^2 u^1 (du^2)^2$

§ 4.3 Angle between two intersecting curves on a surface :



Let c_1 and c_2 be two intersecting curves, intersecting at P on a surface S. If λ^α and μ^β be two vectors in the derection of the tangents to c_1 and c_2 respectively, then $\lambda^\alpha_{(1)} = \frac{du^\alpha}{ds_{(1)}}$, $\mu^\beta_{(2)} = \frac{du^\beta}{ds_{(2)}}$

where $ds_{(1)}$ and $ds_{(2)}$ are the square root of the linear element along u^1-curve and u^2-curve respectively.

Hence the angle θ between the two directions is given by

$$4.10)\quad \cos\theta = a_{\alpha\beta}\frac{du^\alpha}{ds_{(1)}},\frac{du^\beta}{ds_{(2)}}$$

If in particular, the vectors λ^α, μ^β are taken along the parametric curves, then, for the u^1-curve, $du^1 = o$, Consequently 4.5) reduces to $ds^2_{(1)} = a_{11}\left(du^1\right)^2$

and hence the unit vector $\lambda^\alpha_{(1)}$ along u^1 curve is

$$\lambda^\alpha_{(1)} = \left(\frac{du^1}{ds_{(1)}},\frac{du^2}{ds_1}\right) = \left(\frac{1}{\sqrt{a_{11}}},0\right) = \frac{1}{\sqrt{a_{11}}}\delta^\alpha_{(1)}$$

Similarly, the unit vector $\mu^\alpha_{(2)}$ long u^2-curve is $\mu^\alpha_{(2)} = \frac{1}{\sqrt{a_{22}}}\delta^\alpha_{(2)}$

Exercises

1. If θ is the angle between the parametric curves lying on a surface, immersed in E^3, show that

$$\cos\theta = \frac{a_{12}}{\sqrt{a_{11}}\sqrt{a_{22}}} \quad , \quad \sin\theta = \frac{\sqrt{a}}{\sqrt{a_{11}a_{22}}}$$

and hence show that the parametric curves on a surface are orthogonal if and only if $a_{12} = 0$

2. If λ^α, μ^β are two unit vectors such that the rotation of λ^α, μ^β is positive, show that, $\sin\theta = \in_{\alpha\beta}\lambda^\alpha \mu^\beta$

3. Prove that the parametric curves on a surface given by x^1 = a sinu cosv, x^2 = aSin$u$ Sin$v$, x^3 = a cosu form an orthogonal system.

Hints : Find a_{12} and prove that $a_{12} = 0$.

§ 4.4 Geodesic on a surface

Let C be a curve given by c : $\mu^\alpha = \mu^\alpha(t)$

and the length of the curve between the two points P and Q on it be given by

4.11) $s = \int_P^Q \sqrt{a_{\alpha\beta}u^\alpha u^\beta}\, dt, \quad u^\alpha = \frac{du^\alpha}{dt}$

We consider all curves through P and Q. Of all such curves there is in general, one and only one curve whose length from P and Q is less than that of the others. Such a curve is called the geodesic joining P and Q.

Let $\bar{C}$ be any curve in the neighbourhood of C joining P and Q and let it be given by

4.12) $\bar{C} : \bar{u}^\alpha = u^\alpha(t) + \varepsilon\omega^\alpha(t)$

where ω^α is a function of t such that $\omega^\alpha = 0$ at P and Q and ε is a number of infinitesimal order. The are length between P and Q with respect to the curve $\bar{C}$ is given by

$$\bar{s} = \int_P^Q \sqrt{a_{\alpha\beta}\bar{u}^\alpha\bar{u}^\beta}\, dt,$$

We now consider Euler's equation for a functional in V^n.

Let $I = \int_P^Q \phi\left(u^\alpha, u^\alpha\right) dt,$ where ϕ is a function

and $\bar{I} = \int_P^Q \phi\left(u^\alpha + \varepsilon\omega^\alpha, u^\alpha + \varepsilon\omega^\alpha\right) dt,$ by 4.12)

By Taylor's theorem

$\bar{I} = \int_P^Q \phi\left(u^\alpha, u^\alpha\right) dt + \varepsilon \int_P^Q \left(\omega^\alpha \frac{\partial\phi}{\partial u^\alpha} + \omega^\alpha \frac{\partial\phi}{\partial u^\alpha} \right)$ dt, neglecting the other terms

Thus $\bar{I} = I$

$$= \varepsilon \int_P^Q \left(\omega^\alpha \frac{\partial \phi}{\partial u^\alpha} + \omega^\alpha \frac{\partial \phi}{\partial \dot{u}^\alpha} \right) dt$$

$$= \varepsilon \int_P^Q \omega^\alpha \frac{\partial \phi}{\partial u^\alpha} dt + \varepsilon \int_P^Q \frac{\partial \phi}{\partial \dot{u}^\alpha} \dot{\omega}^\alpha dt$$

Integrating the second integral by parts, we get

$$= \varepsilon \int_P^Q \omega^\alpha \frac{\partial \phi}{\partial u^\alpha} dt + \varepsilon \left(\frac{\partial \phi}{\partial \dot{u}^\alpha} \omega^\alpha \right)_P^Q \varepsilon \int_P^Q \frac{d}{dt} \left(\frac{\partial \phi}{\partial \dot{u}^\alpha} \right) \omega^\alpha dt$$

4.13) Hence $\bar{I} - I = \varepsilon \int_P^Q \left\{ \frac{\partial \phi}{\partial u^\alpha} - \frac{d}{dt} \left(\frac{\partial \phi}{\partial \dot{u}^\alpha} \right) \right\} \omega^\alpha dt$, since $\omega^\alpha (P) = \omega^\alpha (Q) = 0$.

If C is a geodesic, then $\bar{I} = I$ must be zero for all neighbouring curves through P and Q i.e. 4.13) must vanish for all arbitrary values of the vector $\omega^\alpha|$ along C. Thus

4.14) $$\frac{d}{dt} \left(\frac{\partial \phi}{\partial \dot{u}^\alpha} \right) - \frac{\partial \phi}{\partial u^\alpha} = 0$$

It is called Euler's or Lagrange's Equation for function ϕ

In our case

$\phi = \sqrt{a_{\beta\gamma} \dot{u}^\beta \dot{u}^\gamma}$ so

$$\frac{\partial \phi}{\partial u^\alpha} = \frac{1}{2\phi} \frac{\partial a_{\gamma\beta}}{\partial u^\alpha} \dot{u}^\beta \dot{u}^\gamma$$

$$\frac{\partial \phi}{\partial \dot{u}^\alpha} = \frac{1}{\phi} a_{\alpha\beta} \dot{u}^\beta$$

substituting in 4.14), we get after a few steps

4.15) $a_{\alpha\beta} \ddot{u}^\beta + [\beta\gamma, \alpha] \dot{u}^\gamma \dot{u}^\beta = \frac{1}{\phi} \frac{d\phi}{dt} a_{\alpha\beta} \dot{u}^\beta$ where we introduce the symbol

4.16) $[\beta\gamma,\alpha]=\frac{1}{2}\left(\frac{\partial a_{\beta\alpha}}{\partial u^{\upsilon}}+\frac{\partial a_{\alpha\lambda}}{\partial u^{\beta}}-\frac{\partial a_{\beta\gamma}}{\partial u^{\alpha}}\right)$

If we define

4.17) $\left\{{\delta \atop \beta\gamma}\right\}=\alpha^{\delta\alpha}[\beta\gamma,\alpha]$

then 4.15) reduces to

$$u^{\delta}+\left\{{\delta \atop \beta\gamma}\right\}u^{\beta}u^{\gamma}=\frac{1}{\phi}.\frac{d\phi}{dt}.u^{\delta}$$

If we choose the parameter t to be the arc length s of the curve i.e. if we set s = t,

then $\phi=\frac{ds}{dt}=1$ *and* $\frac{d\phi}{dt}=0$ and hence we must have

4.18) $\frac{d^2u^{\delta}}{ds^2}+\left\{{\delta \atop \beta\gamma}\right\}\frac{du^{\beta}}{ds}.\frac{du^{\gamma}}{ds}$

which are the desired equations of a geodesic.

Auto-parallel curve : An auto-parallel curve is a curve whose tangent vector field constituted by the tangents at each point of the curve is a parallel vector field.

Exercises

1. Show that the geodesic are the straight lines when the coofdinates are cartesion.
2. Prove that a geodesic is an auto parallel curve.
3. Find the differential equations for the geodesic in
   (a) Spherical and (b) Cylindrical Co-ordinates.
4. Find the geodesic on the surface
   $x^1 = u^1 \cos u^2$, $x^2 = u^1 \sin u^2$, $x^3 = 0$
   inbedded in E^3, the co-ordinates x^i are orthogonal cartesian

   Ans. $\frac{d^2u^1}{ds^2}+\left\{{1 \atop 22}\right\}\left(\frac{du^2}{ds}\right)^2=0$

   $\frac{d^2u^2}{ds^2}+\frac{2}{u^1}.\frac{du^1}{ds}.\frac{du^2}{ds}=0$

5. Find the differential equations of the geodesic for the line element

$ds^2 = (du)^2 + (\sin u)^2 (dv)^2$

Answer : $\frac{d^2u}{ds^2} - \sin u \cos u \left(\frac{dv}{ds}\right)^2 = 0,$

$$\frac{d^2v}{ds^2} + 2\cot u \frac{du}{as} \cdot \frac{dv}{ds} = 0$$

§ 4.5 Gaussion Curvature :

On a surface S, where the metric is $ds^2 = a_{\alpha\beta}\, du_\alpha\, du_\beta$ the Riemann Curvature tensor is given by

$$\frac{\partial}{\partial u^\gamma}\left\{ {\alpha \atop \beta\delta} \right\} - \frac{\partial}{\partial u^\delta}\left\{ {\alpha \atop \beta\gamma} \right\} + \left\{ {\alpha \atop \beta\delta} \right\}\left\{ {\alpha \atop \sigma\gamma} \right\} - \left\{ {\sigma \atop \beta\delta} \right\}\left\{ {\alpha \atop \sigma\delta} \right\}$$

and the associated tensor is given by

$$R_{\alpha\beta\gamma\delta} = a_{\sigma\alpha} R^{\alpha}{}_{\beta\gamma\delta}$$

We recall that, this tensor is skew symmetric in the first two and last two indices. Thus

$$R_{\alpha\beta\gamma\delta} = 0 = R_{\alpha\beta\gamma\gamma}$$

Hence every non vanishing component fo the Riemann tensor is equal to R_{1212} or its negative.

We define a quantity

4.19) $K = \frac{R_{1212}}{a}, \quad a = |a_{\alpha\beta}|$

Such a quantity is called the total curvature of the Gaussian Curvature of the surface. Since only the metric tensor $a_{\alpha\beta}$ and its derivatives are involved in the expression for K, it is an invariant property of the surface.

Note that 4.19 can be written as

4.20) $R_{\alpha\beta\gamma\delta} = K\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}$

Since $\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta} = 2$, we can also write 4.20) as

4.21) $K = \frac{1}{4} R_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}$

Therefore, from equation (4.21) we obtain K is an invariant.

Also we know that $R = a^{\alpha\beta} R_{\alpha\beta}$

$$= a^{\alpha\beta} a^{\alpha\mu} R_{\lambda\alpha\beta\mu}$$

$$= -2K$$

EXERCISES

1. If the co-ordinate system is orthogonal, then show that

$$K = -\frac{1}{2\sqrt{a}}\left[\frac{\partial}{\partial u^1}\left(\frac{1}{\sqrt{a}}.\frac{\partial a_{22}}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{1}{\sqrt{a}}.\frac{\partial a_{11}}{\partial u^2}\right)\right]$$

2. If the line element is of the form

$ds^2 = \left(du^1\right)^2 + a_{22}\left(du^2\right)^2$, show that

$$K = -\frac{1}{\sqrt{a_{22}}}.\frac{\partial^2 \sqrt{a_{22}}}{\left(\partial u^1\right)^2}$$

3. Calculate the Gaussian Curvature for a surface with the metric

i) $ds^2 = a^2 \sin^2 u^1 \left(du^2\right)^2 + a^2\left(du^2\right)^2$

ii) $ds^2 = (du)^2 + \mu^2 (dv)^2$

Answer : i) $K = -\frac{1}{a^2}$

ii) $K = -\frac{1}{\mu}\frac{\partial^2 \mu}{\partial u^2}$

§ 4.6 Developable :

It has already been stated that intrinsic property of a surface depend on the metric tensor of the surface and its derivatives. But the metric of a surface is a local property and it may happen that two surfaces have the same metric.

Consider the following two surfaces :

$$S_1 : \quad y^1 = u^1 \cos u^2 , \quad y^2 = u^1 \sin u^2 , \quad y^3 = a \cosh^{-1} \frac{u^1}{a} .$$

Then $$ds^2 = \frac{\left(u^1\right)^2}{\left(u^1\right)^2 - a^2} \left(du^1\right)^2 + \left(u^1\right)^2 \left(du^2\right)^2 .$$

$$S_2 : \quad y^1 = v^1 \cos v^2 , \quad y^2 = v^1 \sin v^2 , \quad y^3 = a v^2 .$$

Then $$ds^2 = \left(dv^1\right)^2 + \left\{a^2 + \left(v^1\right)^2\right\} \left(dv^2\right)^2 .$$

If we put

$$v^1 = \left\{\left(u^1\right)^2 - a^2\right\}^{1/2} , \quad v^2 = u^2 .$$

then the two surfaces have the some metric. Thus we state :

If two surfaces S_1 and S_2 be such that there exists a coordinate system with respect to which the linear element on S_1 and S_2 are characterised by the same metric tensor, then, they are said to be **isometric** and the transformation of the parameters is called an **isometry.**

A surface which is isometric to the Euclidear plane is called a developable surface or simply a **developable.**

Exercises

1. Prove the $K = 0$ is necessary and sufficient condition for a surface to be a developable.

2. Determine whether the surface with the metric

$ds^2 = (u^2)^2 (du^1)^2 + (u^1)^2 (du^2)^2$ | is a developable or not.

3. Show that the surface defined by

$x^1 = f_1(u^1), \quad x^2 = f_2(u^1), \quad x^3 = u^2$

is a developable where f_1, f_2 are differentiable functions.

§ 4.7 Geodesic Curvature :

Let C be a surface curve defined as

$c : u^\alpha = u^\alpha(s)$

where s is the arc parameter. Then

$$a_{\alpha\beta} \frac{du^\alpha}{ds} \cdot \frac{du^\beta}{ds} = 1$$

Following the line of thought as described in § 3.2 we get $a_{\alpha\beta} \lambda^\alpha \frac{\delta \lambda^\beta}{\delta s} = 0$.

From which it follows that either $\frac{\delta \lambda^\beta}{\delta s}$ is orthogonal to λ^β or $\frac{\delta \lambda^\beta}{\delta s} = 0$.

If $\frac{\delta \lambda^\beta}{\delta s} \neq 0$, we introduce a unit surface vector η codirectional with $\frac{\delta \lambda^\beta}{\delta s}$ | and write

$$\eta = \frac{1}{x_g} \cdot \frac{\delta \lambda^\beta}{\delta s}$$

so that

$$4.24) \quad \frac{\delta \lambda^\beta}{\delta s} = x_g \eta^\beta$$

where x_g is a suitable scalar. This scalar is called the **geodesic curvature** of C. We choose the sense of η such that the rotation $(\lambda^\alpha, \eta^\alpha)$ is positive i.e.

4.25) $\in_{\alpha\beta}\lambda^{\alpha}\eta^{\alpha} = +1$

As

4.26) $\lambda^{\alpha} = \in^{\alpha\beta}\eta_{\alpha}$ and $\eta = \in^{\alpha\beta}\gamma_{\alpha}$

we find that

4.27) $\frac{\delta\eta^{\alpha}}{\delta s} = -x_g\lambda^{\alpha}$ | by 4.24) and 4.25)

We may refer 4.24) and 4.27) as the Frenet formulae for the curve C relative to the surface. It is easy to prove

Theorem : The necessary and sufficient condition for a curve on a surface to be a geodesic is that its geodesic curvature is zero.

Example : We are going to compute the geodesic curvature for a curve

$$C : u^1 = u_0^1, \quad u^2 = \frac{s}{a\cos u_0^1}$$

on the surface of the sphere

$S : x^1 = a\ cosu^1\ cosu^2$

$x^2 = a\ cosu^1\ Sinu^2$

$x^3 = a\ sinu^1$

whose metric is given by

$$ds^2 = a^2\left(du^1\right)^2 + a^2\left(\cos u^1\right)^2\left(du^2\right)^2$$

Then the components of the tangent vector is given by

$$\lambda^{\alpha} = \left(\lambda^1, \lambda^2\right) = \left(\frac{du^1}{ds}, \frac{du^2}{ds}\right) = \left(o, \frac{1}{a\cos u_0^1}\right)$$

The only non vanishing christoffel symbols are

$$\left\{ {1 \atop 22} \right\} = \frac{1}{2a^{11}} \cdot \frac{\partial a_{22}}{\delta u} = \cos u_0^1 \sin u_0^1$$

$$\left\{ {1 \atop 22} \right\} = \frac{1}{2a^{22}} \cdot \frac{\partial a_{22}}{\delta u} = \tan u_0^1 = \left\{ {1 \atop 21} \right\}$$

hence from 4.24)

$$x_g \eta^1 = \frac{\delta \lambda^1}{\delta s} = \frac{d\lambda^1}{ds} + \left\{ {1 \atop 22} \right\} \lambda^2 \lambda^2 = \frac{1}{a^2} \tan u_0^1$$

$$x_g \eta^2 = 0$$

But

$$a_{\alpha\beta} \eta^\alpha \eta^\beta = +1 \qquad \therefore x_g = \frac{\tan u_0^1}{a}$$

Exercises

1. Consider the surface of the right circular cone

$$S: \ x^1 = u^1 \cos u^2$$
$$x^2 = u^1 \sin u^2$$
$$x^3 = u^1$$

and the curve C whose equation are taken in the form

$$C: \ u^1 = a$$

$u^2 = \frac{s}{a}$, where s is the arc parameter show that

$$x_g = \frac{\sqrt{2}}{2a}$$

2. Show that the geodesic curvature of the curve u = c on a surface with metric $\phi^2 du^2 + \mu^2 (d\nu)^2$ is $\frac{1}{\phi u} \cdot \frac{\partial \mu}{\partial u}$

3. Prove that

(i) $x_g^2 = a_{\alpha\beta} \frac{\delta \lambda^\alpha}{\delta s} \cdot \frac{\delta \lambda^\beta}{\delta s}$

(ii) $k_g = \in_{\alpha\beta} \lambda^{\alpha} . \frac{\delta\lambda^{\beta}}{\delta s}$

4. Deduce that the geodesic curvature $x_g^{(1)}$ of the u^1 curve and $x_g^{(2)}$ of the u^2-curve are given by

$$k_g^{(1)} = \frac{1}{\sqrt{a}} . \frac{\partial}{\partial u^1}\left(\frac{a_{12}}{\sqrt{a_{11}}}\right) - \frac{1}{\sqrt{a}} . \frac{\partial\sqrt{a_{11}}}{\partial u^1}$$

$$k_g^{(2)} = \frac{1}{\sqrt{a}} . \frac{\partial}{\partial u^2}\left(\frac{a_{12}}{\sqrt{a_{22}}}\right) - \frac{1}{\sqrt{a}} . \frac{\partial\sqrt{a_{22}}}{\partial u^1}$$

Hence show that, when the co-ordinate curves are orthogonal

$$k_g^{(1)} = \frac{1}{2\sqrt{a_{22}}} . \frac{\partial(\log a_{11})}{\partial u^2}, \quad k_g^{(2)} = \left(\frac{1}{2\sqrt{a_{11}}}\right) . \frac{\partial(\log a_{22})}{\partial u^1}$$

5. Show that the conditions that the u^1-curve and the u^2-curve be geodesic are $\left\{ {2 \atop 11} \right\} = 0$ and $\left\{ {1 \atop 22} \right\} = 0$ respectively.

§ 4.8 The tangent vector and the normal vector to a surface :

We are now going to investigate the properties of a surface in its relation to the surrounding space. Consequently we are dealing with two distinct systems of co-ordinates, namely, the three curvilinear co-ordinates for the surrounding space which we denote by $x^{i(i=1,2,3)}$ and the two curvilinear co-ordinates of the surface which we denote by $u^{\alpha}(\alpha=1,2)$. There will now be tensors of the surrounding space and also tensors of the surface.

Let $x^i = x^i\left(u^1, u^2\right)$

be the equation of a suface S imbedded in E^3. If we take a small displacement du^α on the surface, the corresponding component of the displacement in space are given by

4.28) $dx^i \dfrac{\partial x^i}{\delta u^\alpha} du^\alpha$

Now du^i is a space vector and is surface invariant i.e. its components are unaltered if the Gaussion co-ordinates alone are transformed. Similarly du^α is a surface vector and is also a space invariant. Hence if we regard 4.28) first from the point of view of a transformation of space co-ordinates and then from the point of view of a transformation of surface co-ordinates, we see that $\dfrac{\partial x^i}{\delta u^\alpha}|$ is a contravariant space vector and also a covariant surface vector, so that, it may be represented by

4.29) $x^i_\alpha = \dfrac{\partial x^i}{\delta u^\alpha}$

The line element in space is given by

$ds^2 = g_{mn} dx^m dx^n$

and that on the surface is given by

$ds^2 = a_{\alpha\beta} du^\alpha du^\beta$

using 4.28) and 4.29) we get

4.30) $a_{\alpha\beta} = g_{mn} x^m_\alpha x^n_\beta$

From 4.28) we find that

$\dfrac{dx^i}{ds} = \dfrac{\partial x^i}{\partial u^\alpha} . \dfrac{du^\alpha}{ds}$ i.e.

4.31) $\lambda^i = x^i_\alpha \lambda^\alpha$

This formula tells us that any surface vector A^α can be viewed as a space vector with components A^i determined by

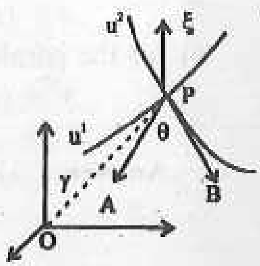
4.32) $A^i = x^i_\alpha A^\alpha$

We shall refer to a vector A^i determined by this formula as a **tangent vector to the surface S.**

Let A and B be two surface vectors drawn at P on S such that the rotation A, B is positive. The unit normal vector ξ to the surface S is orthogonal to both A and B and is so oriented that (A, B, ξ) form a right handed system i.e.

4.33) $\in_{ijk} = A^i B^j \xi^k = +1$

4.34) $\xi = \frac{A\times B}{|A\times B|} = \frac{A\times B}{|A||B|Sin\theta}$ where θ is the angle between them



As the point P(u^1 u^2) is displaced to a new position $Q(u^1+du^1, u^2+du^2)$, the vector ξ undergoes a change

4.35) $d\xi = \frac{\partial\xi}{\partial u^\alpha} du^\alpha = \xi du^\alpha$, say whereas the position vector r is changed to the amount

4.36) dr= $r_\beta du^\beta$, say, then

4.37) $d\xi.dr = \xi_\alpha . r_\beta du^\alpha du^\beta$

if we define

4.38) $b_{\alpha\beta} = -\frac{1}{2}(\xi_\alpha r_\beta + \xi_\beta r_\alpha)$

then 4.37) becomes

4.39) $d\xi.dr = b_{\alpha\beta} du^\alpha du^\beta$

The left hand side, being the scalar product of two vectors, is an invariant. From the quotient law, $b_{\alpha\beta}$ is a covariant tensor of order (0,2) and is symmetric from 4.38)

The quadratic form

4.40) $B = b_{\alpha\beta} du^{\alpha} du^{\beta}$

introduced by Gauss, is called the **second fundamental form** of surface. It plays an important part in the study of surfaces when they are viewed from the surrounding space.

Exercises

1. Calculate the second fundamental form
   i) for the right helicoid given by
   $r = (u\cos v,\ u \sin v,\ cv)$
   ii) for the paraboloid given by
   $r = (u,\ v,\ u^2 - v^2)$

Answer i) $B = -\dfrac{2c}{(u^2 + c^2)^{1/2}}$

ii) $B = -\dfrac{2\left[(du)^2 - (dv)^2\right]}{(4u^2 + 4v^2 + 1)^{1/2}}$

§ 4.9 Tensor Derivative :

Consider a curve C lying on a given surface S, with parameter T. If A^i is a space vector, defined along C, we can compute the intrinsic derivative of A^i namely

$$\frac{\delta A^i}{\delta t} = \frac{dA^i}{dt} + {}_{g}\left\{ {}^{i}_{jk} \right\} A^j \frac{dx^k}{dt}$$

where ${}_{g}\left\{ {}^{i}_{jk} \right\}$ refer to the space co-ordinates x^i and are formed from the metric co-efficient g_{ij}

Again, if we consider a surface vector A^{α} defined along C, we can form the intrinsic derivative with respect to the surface variables, namely

$$\frac{dA^\alpha}{dt}=\frac{dA^\alpha}{dt}+_a\left\{ {\alpha \atop \beta\gamma} \right\} A^\beta \frac{du^\gamma}{dt}$$

where $_a\left\{ {\alpha \atop \beta\gamma} \right\}$ are formed from the metric co-efficients $a_{\alpha\beta}$ associated with the surface coordinates u^α

A geometric interpretation of these formulas is at hand where A^i and A^α are such that $\frac{\delta A^i}{\delta t}=0$ and $\frac{\delta A^\alpha}{\delta t}=0$

In the first case, the vectors A^i form a parallel field with respect to C, considered as a space curve, whereas in the latter case, the vectors A^α form a parallel field with respect to C regarded as a surface curve.

Consider next a tensor field L^i_α which is a contravariant vector with respect to a transformation of spece co-ordinates x^i and a covariant vector relative to a transformation of surface coordinates u^α. If L^i_α is defined along C, with t as a parameter, then L^i_α is a function of t. We form an invariant $\varphi(t)=L^i_\alpha A_i B^\alpha$

where A_i, B^α are the parallel vector field along C, as explained earlier.

Then we must have $\frac{d\varphi(t)}{dt}=\left(\frac{dL^i_\beta}{dt}+_g\left\{ {i \atop jk} \right\} L^j_\alpha \frac{dx^k}{dt}-_a\left\{ {\alpha \atop \beta\gamma} \right\} L^i_\alpha \frac{du^\gamma}{dt}\right)A_i B^\beta$

By Quotient Law, the terms within the bracket is a tensor. Such a tensor is called the **intrinsic derivative of** L^i_α with respect to t and we write

$$4.41)\quad \frac{\delta L^i_\beta}{\delta t}=\frac{\partial L^i_\alpha}{\partial t}+_a\left\{ {i \atop jk} \right\} L^j_\alpha \frac{dx^k}{dt}-\left\{ {\alpha \atop \beta\gamma} \right\} L^i_\delta \frac{du^\gamma}{dt}$$

$$=\left(\frac{\partial L^i_\beta}{\partial t^\gamma}+_g\left\{ {i \atop jk} \right\} L^j_\alpha \frac{\partial x^k}{\partial y^\gamma}-_a\left\{ {\alpha \atop \beta\gamma} \right\} L^i_\alpha\right)\frac{du^\gamma}{dt}$$

Since $\frac{du^\gamma}{dt}$ is an arbitraty surface vector, we conclude that

4.42) $$L^i_{\alpha\gamma} = \frac{\partial L^i_\alpha}{\partial y^\gamma_u} +_g \left\{ {i \atop jk} \right\} L^j_\alpha x^k_\gamma -_a \left\{ {\delta \atop \alpha\gamma} \right\} L^i_\delta$$

is a tensor. We shall call $L^i_{\alpha.\gamma}$, the tensor derivative of L^i_α with respect to u^γ

Let us take the tensor derivative of x^i_α defined in 4.29).

We find that

$$x^i_{\alpha,\beta} = \frac{\partial x^i}{\partial u^\alpha \partial u^\beta} +_g \left\{ {i \atop jk} \right\} \frac{\partial x^j}{\partial u^\alpha} . \frac{\partial x^k}{du^\beta}{}_a - \left\{ {\gamma \atop \alpha\beta} \right\} \frac{\partial x^i}{\partial u^\gamma}$$

from which it follows that $x^i_{\alpha.\beta} = x^i_{\beta,\alpha}$

Fròm the equation $a_{\alpha\beta} = g_{mn} x^m_\alpha x^n_\beta$

We find that

4.43) $g_{mn} x^m_{\alpha,\gamma} x^n_\beta + g_{mn} x^m_\alpha x^n_{\beta,\gamma} = 0$

Interchanging α, β, γ cyclically, we get

4.44) $g_{mn} x^m_{\beta,\alpha} x^n_\gamma + g_{mn} x^m_\beta x^n_{\alpha,\gamma} = 0$

4.45) $g_{mn} x^m_{\gamma,\beta} x^n_\alpha + g_{mn} x^m_\gamma x^n_{\alpha,\beta} = 0$

Adding 4.44) and 4.45) and subtracting 4.43) we get,

4.46) $g_{mn} x^m_{\alpha,\beta} x^n_\gamma = 0$

This result, interpreted geometrically, means that $x^i_{\alpha,\beta}$, from the point of view of the space coordinates is a space vector orthogonal to the surface and hence it is codirectional with the normal vector ξ^i. Consequently there must exist quantities $b_{\alpha\beta}$ such that

4.47) $x^i_{\alpha,\beta} = b_{\alpha\beta}\xi^i$

The equivalence of this definition, of the tensor $b_{\alpha\beta}|$ that given in 4.38) can be proved after a brief calculation.

Such equations are known as ***Gauss's formula.***

Note that

$$g_{ij}\xi^i\xi^j = 1$$

and hence

$$g_{ij},\xi^i_{,\alpha}\xi^j = 0$$

This equation shows that $\xi^i_{,\alpha}|$ is orthogoral to the unit normal ξ^i and hence it lies in the tangent plane of the surface. Accordingly, it can be expressed as

4.48) $\xi^i_{,\alpha} = c^\beta_\alpha x^i_\beta$

where c^β_α,s are to be determined

As ξ^i is normal to the surface $g_{ij}\xi^i x^j_\alpha = 0$

Taking its tensor derivative and using 4.47) and 4.48 we get

$$a_{\gamma\alpha}c^\gamma_\beta + b_{\alpha\beta} = 0$$

4.49) $c^\gamma_\beta = -a^{\alpha\delta}b_{\alpha\beta}$

consequently 4.48) reduces to

4.50) $\xi^i_{,\alpha} = -a^{\alpha\delta}b_{\alpha\beta}x^i_\beta$

These equations are known as **Weingarten's formula**

If we write

4.51) $c_{\alpha\beta} = g_{ij}\xi^i_{,\alpha}\xi^j_{,\beta}$

we see that $c_{\alpha\beta}$ is symmetric tensor of order (0,2) and we call the quadratic form

4.52) $C = c_{\alpha\beta} du^\alpha du^\beta$

the **third fundamental form** of the surface.

Exercises

1. Show that

$$b_{\alpha\beta} = g_{ij} x^i_{,\alpha\beta} \xi^j = \frac{1}{2} \in^{\gamma\delta} \in_{ijk} x^i_{\alpha,\beta} x^j_\gamma x^k_\delta$$

2. If the space coordinates are rectangular Cartesian, show that

$$b_{\alpha\beta} = \frac{1}{\sqrt{a}} \in_{ijk} \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} x^j_1 x^k_2$$

3. If $\beta^{\gamma\delta}$ is the cofactor of $b_{\gamma\delta}$ in $|b_{\gamma\delta}|$, divided by $|b_{\gamma\delta}|$, show that,

$$x^r_\alpha = -a_{\alpha\delta} p^{\gamma\delta} \xi^r_{,\gamma}$$

4. Prove that $a^{\alpha\beta} x^r_{\alpha,\beta} = 2H\xi^r$

where

4.53) $H = \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta}$ is called the **mean curvature** of the surface

5. $b_{\alpha\beta} = 0$ identically, then show that the surface is a plane.
6. Show that a surface is a sphere if and only if the second fundamental form is a non-zero constant multiple of its first fundamental form.
7. Find the mean curvature of the surface

i) $x^1 x^2 = x^3$

ii) $r = (u, v, u^2 - v^2)$

§ 4.10 The equations of Gauss and Codazzi :

Taking the tensor derivative of the equation 4.47) we get

$$x^i_{\alpha,\beta\gamma} = b_{\alpha\beta,\gamma}\xi^i + b_{\alpha\beta}\xi^i{}_{,\gamma}$$

Using 4.50) we get

$$x^i_{\alpha,\beta\gamma} = b_{\alpha\beta,\gamma}\xi^i - b_{\alpha\beta}a^{\mu\sigma}b_{\mu\gamma}x^i_\sigma$$

$$\therefore \quad x^i_{\alpha,\beta\gamma} - x^i_{\alpha,\gamma\beta} = \left(b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta}\right)\xi^i + b_{\alpha\gamma}a^{\mu\sigma}b_{\mu\beta}x^i_\sigma - b_{\alpha\beta}a^{\mu\sigma}b_{\mu\gamma}x^i_\sigma$$

$$= \left(b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta}\right)\xi^i + \left(b_{\alpha\gamma}a^{\mu\sigma}b_{\mu\beta}x^i_\sigma - b_{\alpha\beta}a^{\mu\sigma}b_{\mu\gamma}\right)x^i_\sigma$$

Applying Ricci identity

$$x^i_{\alpha,\beta\gamma} - x^i_{\alpha,\gamma\beta} = R^\sigma_{\alpha\beta\gamma}x^i_\sigma$$

where $R^\sigma_{\alpha\beta\gamma}$ is the Reimann Curvature tensor of the surface, then (4.53)

4.53) $R^\sigma_{\alpha\beta\gamma}x^i_\sigma = \left(b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta}\right)\xi^i + \left(b_{\alpha\gamma}a^{\mu\sigma}b_{\mu\beta} - b_{\alpha\beta}a^{\mu\sigma}b_{\mu\gamma}\right)x^i_\sigma$

Multiplying 4.53) by ξ_i and using the fact that

$$\xi^i\xi_i = 1, x^i_\sigma\xi_i = 0$$

we find from above

4.54) $b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta} = 0$

These equations are called the **Codazzi Equations** of the surface.

Again, multiplying 4.53) by $g_{ij}x^j_p$ we get

$$b_{\alpha\beta,\gamma}R^\sigma{}_{\alpha\beta\gamma} = a_{\rho\sigma}b_{\alpha\gamma}a^{\mu\sigma}b_{\sigma\beta} - a_{\rho\sigma}b_{\alpha\beta}a^{\mu\sigma}b_{\mu\gamma}$$

Therefore,

4.55) $R_{\rho\alpha\beta\gamma} = b_{\rho\beta}b_{\alpha\gamma} - b_{\rho\gamma}b_{\alpha\beta}$

These equations are called the **Gauss's Equation** of the surface.

It is known that the only non-zero components of $R_{\rho\alpha\beta\gamma}$ are R_{1212} and its negative. Thus there are only two independent Codazzi Equations and only one independent Gauss's equation, namely

4.56) $b_{11,2} - b_{12,1} = 0$ and $b_{22,1} - b_{21,2} = 0$

and

4.57) $R_{1212} = b_{11}b_{22} - b^2 = b$ where $b = |b_{ij}|$

Hence from 4.19) we see that

4.58) $K = \dfrac{b}{a}$

If at every point of a surface

4.59) $H = \dfrac{1}{2} a^{\alpha\beta} b_{\alpha\beta} = 0$

we say the surface is **minimal** and H is called the mean curvature.

Exercises

1. Show that the right helicoid given by

$$y^1 = u^1 \cos u^2, \quad y^2 = u^2 Sin u^2, \quad y^3 = cu^2$$

is a minimal surface.

2. Prove that

C – 2HB + K A = 0

where the notations have their usual meanings

3. Show that

$$a^{\alpha\beta} C_{\alpha\beta} = 4H^2 - 2K$$

4. If λ^α is a unit vector of the surface, show that

$$K = 2Hb_{\alpha\beta}\lambda^\alpha\lambda^\beta - c_{\alpha\beta}\lambda^\alpha\lambda^\beta$$

5. Find the Gaussian Curvature at the point (2, 0, 1) of the surface

$$(x^1)^2 - (x^2) = 4x^3$$

where x^1, x^2, x^3 are rectangular Cartesion co-ordinates. $\left[\text{Answer} := -\frac{1}{16}\right]$

6. Show that

i) $g_{mn}\xi^m\xi^n{}_{,\alpha\beta} = -c_{\alpha\beta}$

ii) $\xi^r{}_{,\alpha\beta} = -c_{\alpha\beta}\xi^r - a^{\sigma\mu}b_{\sigma\alpha,\beta}x^\gamma_\mu$

iii) $a^{\alpha\beta}\xi^r{}_{,\alpha\beta} = -(4H^2 - 2K)\xi^r - 2a^{\alpha\beta}\frac{\partial H}{\partial u^\alpha}x^r_\beta$

§ 4.11 The Curves on a surface

The equation of a surface is given by

$$S : x^i = x^i(u^1, u^2)$$

and the equation of a curve c lying on S is given by

$$c : u^\alpha = u^\alpha(s)$$

Hence

$$x^i = x^i(u^1(s), u^2(s) = x^1(s)$$

is the equation of c regarded as a space curve

Taking the intrinsic derivative of both sides of 4.31) we get

$$\frac{\delta\lambda^i}{\delta s} = x^i_{\alpha,\beta}\frac{du^\beta}{ds}\lambda^\alpha + x^i_\alpha\frac{\delta\lambda^\alpha}{\delta s}$$

or $x\mu^i = b_{\alpha\beta}\xi^i\lambda^\beta\lambda^\alpha + x^i_\alpha x_g\varsigma^\alpha$

4.60) $x\mu^i = b_{\alpha\beta}\lambda^\alpha\lambda^\beta\xi^i + x_g\zeta^\alpha$

Let θ be the angle between the principal normal μ^i and the surface normal ξ^i.

Then cos $\theta = g_{ij}\xi^i\mu^j$

Hence from 4.60) we find that

$$x g_{ij}\xi^i\mu^j = b_{\alpha\beta}\lambda^\alpha\lambda^\beta g_{ij}\xi^i\xi^j + x_g\xi^j g_{ij},\xi^i$$

4.61) $x \quad \cos\theta = b_{\alpha\beta}\lambda^\alpha\lambda^\beta$

From 4.61) we see that the quantity $b_{\alpha\beta}\lambda^\alpha\lambda^\beta|$ is the same for all curves of the surface which have the same tangent vector λ^α and consequently $x\cos\theta|$ is also the same for all such curves. We have therefore

Meusnier's Theorem : For all curves on a surface which have the same tangent vector, the quantity $x\cos\theta$ has the same value, where θ is the angle between the principal normal and the surface normal at a point on the curve whose curvature is k.

The quantity x $\cos\theta$ is called the **normal curvature** of the surface in the direction λ^α and we write it as

$$x_{(n)} = b_{\alpha\beta}\lambda^\alpha\lambda^\beta$$

$$= b_{\alpha\beta}\frac{du^\alpha}{ds}.\frac{du^\beta}{ds}$$

$$= \frac{b_{\alpha\beta}du^\alpha du^\beta}{ds^2}$$

4.62) $$k_{(n)} = \frac{b_{\alpha\beta}du^\alpha du^\beta}{a_{\alpha\beta}du^\alpha du^\beta}$$

Exercises

1. Find the normal curvature of the right helicoid.
2. Prove that the normal curvature in the direction of the co-ordinate curves are b_{11}/a_{11} and b_{22}/a_{22}
3. It a curve is a geodesic on the surface, prove that, it is either a straight line or its principal normal is orthogonal to the surface at every point and conversely.

$$\left[Ans. x_{(n)} = -\frac{2c\,du\,dv}{\sqrt{u^2+c^2}\left\{(du)^2+(u^2+c^2)\right\}} \right]$$

§ 4.12Principal Curvatures :

From 4.62) we that the normal curvature at a point depends on the directions du^1, du^2. We shall now find out the directions du^1, du^2 for which $x_{(n)}$ has the extreme values. These directions are called the **principal directions** at the given point and the corresponding values for $x_{(n)}$ are called the **principal curvatures.** Let us denote $k_{(n)}$ by $k_{(p)}$ for the principal curvatures.

Now 4.62) can be written as

$$b_{\alpha\beta}du^\alpha du^\beta - k_{(p)}a_{\alpha\beta}du^\alpha du^\beta = 0$$

Differentiating with respect to u^α we get for the principal directions

4.63) $\left(b_{\alpha\beta} - x_{(p)}a_{\alpha\beta}\right)\lambda^\beta = 0$

The above set of homogeneous equations will possess non-trivial solutions for λ^β if and only if the value of $x_{(p)}$ are the roots of the determinant equations.

$$\left|b_{\alpha\beta} - k_{(p)}a_{\alpha\beta}\right| = 0$$

or $\quad k_{(p)^2} - k_{(p)}\dfrac{a_{11}b_2 + a_{11}b_{22} - 2a_{12}b_2}{a} + \dfrac{b}{a} = 0$

4.64) $k_{(p)^2} - 2Hx_{(p)} + K = 0$

We shall denote the two roots of $k_{(p)}$ by $k_{(1)}$ and $k_{(2)}$ and call them the principal curvatures of the surface and the directions corresponding to $k_{(1)}$ and $k_{(2)}$ are called the principal directions on the surface. Thus

4.65) $k_{(1)} + k_{(2)} = 2H$

and $k_{(1)}k_{(2)} = K$

From 4.63), it follows that the principal directions $\lambda_{(1)}\ \lambda_{(2)}$, say, corresponding to $x_{(1)}$ and $x_{(2)}$ respectively are determined by

4.66) $$\begin{cases}\left(b_{\alpha\beta}-x_{(1)}a_{\alpha\beta}\right)\lambda^{\beta}_{(1)}=0\\ \left(b_{\alpha\beta}-x_{(2)}a_{\alpha\beta}\right)\lambda^{\beta}_{(2)}=0\end{cases}$$

Multiplying the first of 4.66) by the $\lambda^{\beta}_{(2)}$ and the second by $\lambda^{\alpha}_{(1)}$ and then subtracting we get

$$\left(x_{(2)}-x_{(1)}\right)a_{\alpha\beta}\lambda^{\alpha}_{(1)}\lambda^{\beta}_{(2)}=0$$

A point at which $x_{(1)}=x_{(2)}$ is called on **umbilic point.** Thus at all other points

$$a_{\alpha\beta}\lambda^{\alpha}_{(1)}\lambda^{\beta}_{(2)}=0$$

and we state

Theorem : At each non-umbilical point of a surface there exist two multually orthogonal directions for which the normal curvature attains its extreme values.

A curve on a surface such that the tangent line to it at every point is directed along a principal directions is called **a line of curvature.**

From 4.63) we get

$$b_{\alpha\beta}\lambda^{\beta}=x_{(p)}a_{\alpha\beta}\lambda^{\beta}$$

when $\alpha=1\quad b_{1\beta}\lambda^{\beta}=x_{(p)}a_{1\beta}\lambda^{\beta}$

when $\alpha=2\quad b_{2\beta}\lambda^{\beta}=x_{(p)}a_{2\beta}\lambda^{\beta}$

Thus eliminating $x_{(p)}$ we get

$$\frac{b_{1\beta}du^{\beta}}{a_{1\beta}du^{\beta}}=\frac{b_{2\beta}du^{\beta}}{a_{2\beta}du^{\beta}}$$

4.67) $(b_{11}a_{12}-b_{12}a_{11})(du^1)^2+(b_{11}a_{22}-b_{22}a_{11})du^1du^2$

$$+(b_{12}a_{22}-b_{12}a_{22})(du^1)^2=0$$

The above equation is the equation of the lines of curvature of the surface.

Exercises

1. Show that the parametric curves are the lines of curvature if and only if $a_{12}=b_{12}=0$
2. Show that the lines of curvature on a minimal surface form an isometric system.
3. Prove that the lines of curvature on a surface are given by

$$\varepsilon^{\gamma\delta}a_{\alpha\gamma}b_{\beta\delta}du^{\alpha}du^{\beta}=0$$

A surface is called a **surface of positive curvature** if at all points, the Gaussian Curvature K is positive.

A point on a surface is called **elliptic** if $K>0$

A surface is called a **surface of negative curvature** if at all points, the Gaussian curvature K is negative.

A point on a surface is called hyperbolic if $K<0$

A point on a surface is called parabolic if $K=0$

Example : The right helicoid is a surface of negative curvature.

The directions on the surface given by the equation $b_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}=0$

are called the **asymptotic directions** and the curves whose tangents are asymptotic directions are called asymptotic lines of the surface.

Exercises

1. Show that the torsion of an asymptotic line equals $\pm\sqrt{-K}$ where K is the Gaussian curvature of the surface.

2. Prove that the parametric curves on a surface are asymptotic lines is and only if $b_{11} = b_{22} = 0$ and show that

$$K = -\frac{b_{12}^2}{a}, \quad H = -\frac{a_{12}b_{12}}{a}, \quad \frac{a_{12}}{b_{12}} = \frac{H}{K}$$

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PG (MT) 08 : Group B

Unit 1 □ Graphs and Digraphs

Structure

1.1 Graph

1.2 Directed Graph

1.3 Worked Out Exercises

1.4 Exercises

1.1 Graph

We define a **graph** G (undirected) as a triple (V, E, γ), where

(i) V is a finite non-empty set whose elements are called **vertices**,

(ii) E is a finite set (may be empty) whose elements are called **edges**, and

(iii) γ is a function, known as **incidence function**, which assigns to each edge an unordered pair $\{v_i, v_j\}$ of vertices.

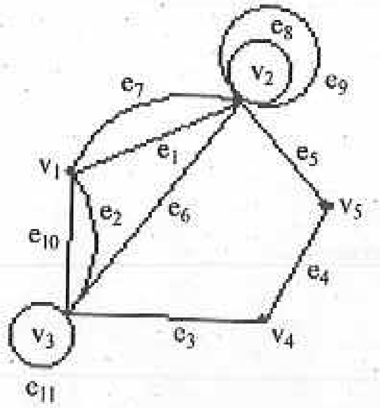
Observe that v_i and v_j may be the same and when E is empty, γ is empty. Some authors use the notation (V, E) to denote a graph with the tacit understanding that every edge is associated with two vertices (may be the same).

Let G = (V, E, γ) be a graph and $\gamma(e_k) = \{v_i, v_j\}$. Then we say that the edge e_k is **associated** with the vertices v_i, v_j and that v_i and v_j are adjacent. v_i, v_j are called the end points or, end vertices of the edge e_k. A vertex v_i and an edge e_k are said to be incident with each other if v_i is an end point of e_k.

Geometrical representation of graphs : Generally graphs are represented by means of a diagram in which the vertices are represented as points or, small circles and each edge as a line (may not be straight) joining the adjacent vertices.

Example 1.1.1 Consider the graph G = (V, E, γ) where V = $\{v_1, v_2, v_3, v_4, v_5\}$, E = $\{e_1, e_2, \ldots, e_{11}\}$ and γ is defined by $\gamma(e_1) = \gamma(e_7) = \{v_1, v_2\}$, $\gamma(e_2) = \gamma(e_{10}) = \{v_1, v_3\}$, $\gamma(e_3) = \{v_3, v_4\}$, $\gamma(e_4) = \{v_4, v_5\}$, $\gamma(e_5) = \{v_2, v_5\}$, $\gamma(e_6) = \{v_2, v_3\}$, $\gamma(e_8) = \gamma(e_9) = \{v_2, v_2\}$, $\gamma(e_{11}) = \{v_3, v_3\}$.

This graph may be represented by the following diagram :



G

Fig. 1.1.1

Observe that the diagram may be drawn in different ways. The way the diagram is drawn is not important. In the diagram we only show the vertices and the edges and the incidence relation between them.

For a graph G we now define the following terms :

Parallel edges : If two or more edges are associated with the same pair of distinct vertices, then these edges are called **parallel edges**. In the graph in fig. 1.1.1, the edges e_1, e_7, and e_2, e_{10} are parallel edges, since $\gamma(e_1) = \gamma(e_7) = \{v_1, v_2\}$ and $\gamma(e_2) = \gamma(e_{10}) = \{v_1, v_3\}$.

Adjacent edges : Two non-parallel edges are said to be **adjacent** if they are incident with a common vertex.

In the graph in fig. 1.1.1, the edges e_1 and e_2 are adjacent since they are incident with the common vertex v_1. But e_1 and e_7 are not adjacent since they are parallel edges. Of course, there are several other adjacent edges in the graph in fig. 1.1.1.

Loop : If e_j be an edge such that $\gamma(e_j) = \{v_i, v_i\}$, then e_j is called a **loop** (or, self-loop) on the vertex v_i. In this case, the edge e_j has the same vertex as both its end points.

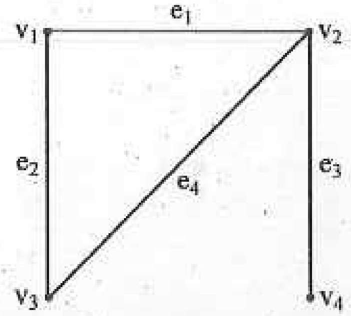
In the graph in fig. 1.1.1, the edges e_8, e_9, e_{11} are all loops. Both the loops e_8, e_9, are on the vertex v_2 whereas the loop e_{11} is on the vertex v_3.

Isolated Vertex : A vertex is called an **isolated vertex** if it is not adjacent to any vertex.

The graph in fig. 1.1.1 does not contain any isolated vertex.

Simple graph : A graph that does not contain any loop and any parallel edge is called a **simple graph.**

Example 1.1.2 Consider the graph $G = (V, E, \gamma)$ where $V = \{v_1, \ldots, v_4\}$, $E = \{e_1, \ldots, e_4\}$ and $\gamma(e_1) = \{v_1, v_2\}$, $\gamma(e_2) = \{v_1, v_3\}$, $\gamma(e_3) = \{v_2, v_4\}$, $\gamma(e_4) = \{v_2, v_3\}$. G can be represented as :

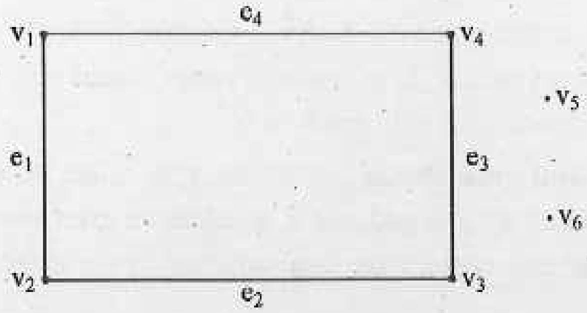


G

Fig. 1.1.2

G does not contain any loop and any parallel edge. Hence it is a simple graph.

Example 1.1.3. Let $G = (V, E, \gamma)$ be a graph, where $V = \{v_1, \ldots, v_6\}$, $E = \{e_1, \ldots, e_4\}$ and $\gamma(e_1) = \{v_1, v_2\}$, $\gamma(e_2) = \{v_2, v_3\}$, $\gamma(e_3) = \{v_3, v_4\}$, $\gamma(e_4) = \{v_4, v_1\}$. Then G can be represented as in fig. 1.1.3.



G

Fig. 1.1.3

We see that G does not contain any parallel edge or any loop. Also, the vertices v_5 and v_6 are not adjacent to any vertex. Hence G is a simple graph with two isolated vertices v_5 and v_6.

Null Graph : If a graph $G = (V, E, \gamma)$ be such that $E = \phi$ (empty), then G is called a **null graph**.

For example, the following graph in fig. 1.1.4 is a null graph.

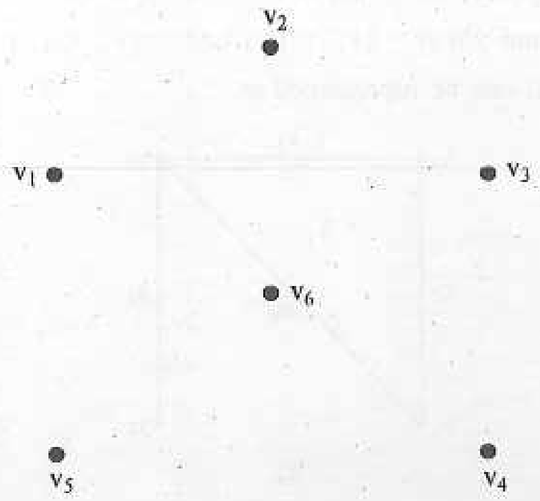


Fig. 1.1.4

Observe that in a null graph every vertex is an isolated vertex.

Degree of a vertex : Let G be a graph and v_i, a vertex of G. The **degree** of v_i, denoted $d(v_i)$, is the number of edges incident with v_i, where a loop at v_i is assumed to contribute two to the degree of v_i.

For example, in the graph 1.1.1, $d(v_1) = 4$, $d(v_2) = 8$, $d(v_3) = 6$, $d(v_4) = 2$, $d(v_5) = 2$. Observe that a vertex v_i is an isolated vertex if and only if $d(v_i) = 0$. Thus, the degree of every vertex in a null graph is 0.

Odd vertex and even vertex : A vertex v_i is called an **odd degree vertex** or, simply **odd vertex** if $d(v_i)$ is odd and it is called an **even vertex** if $d(v_i)$ is even. In the graph 1.1.1 all the vertices are even whereas in the graph 1.1.2, the vertices v_2 and v_4 are odd.

Pendant Vertex (or, end vertex) : Let v_i be a vertex of a graph G. v_i is called a **pendant vertex** or, an end vertex if $d(v_i) = 1$. In the graph 1.1.2 the vertex v_4 is a pendant vertex.

Degree-sequence of a graph : If we list the degrees of all the vertices of a graph G in non-decreasing order, we get the **degree-sequence** of G. For example, the graph

1.1.1 has the degree-sequence (2,2,4,6,8). Different graphs may have the same degree-sequence. For example, the following two graphs G_1 and G_2 are different even though they have the same degree-sequence (2, 2, 2, 2, 2, 2, 2).

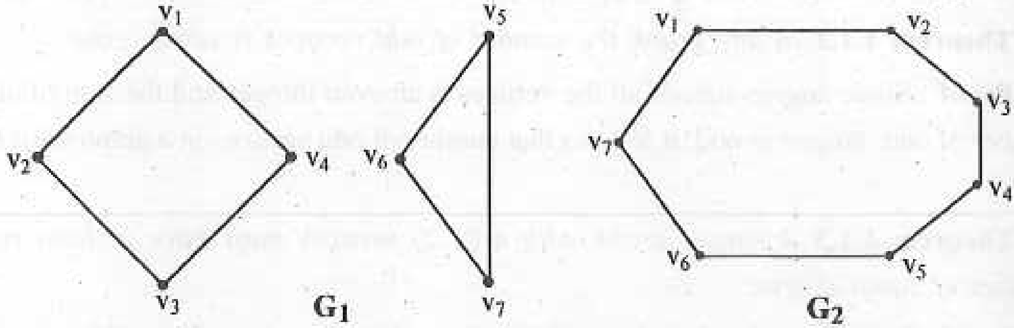


Fig. 1.1.5

Regular graph : A graph G is called a **regular graph** if all its vertices are of equal degree. If the degree of each vertex of G is p, then G is called a p-regular graph. The following graph in fig. 1.1.6 is a 3-regular graph.

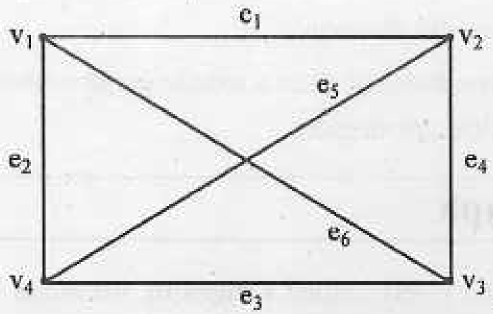


Fig. 1.1.6 (3-regular graph)

Basic Properties of a graph : We now consider a few basic properties of a graph in the form of the following theorems.

Theorem 1.1.1 : *The sum of the degrees of all the vertices in any graph G is twice the number of edges in G.*

Proof : Let G be a graph with n edges. If e is a loop on the vertex v_i, then it contributes two to the degree of v_i. If e is incident with two distinct vertices v_i and v_j, then e contributes two to the degree-sum, one for the $d(v_i)$ and one for the $d(v_j)$. Hence, the sum of the degrees of all the vertices in G is 2n i.e., twice the number of edges in G.

This result is known as the handshaking lemma.

Corollary 1.1.1 The sum of the degrees of all the vertices in any graph is an even non-negative integer.

Corollary 1.1.2 If G is a p-regular graph with n vertices, then G has $\frac{1}{2}$pn edges.

Theorem 1.1.2 *In any graph the number of odd vertices is alway even.*

Proof : Since degree-sum of all the vertices is an even integer and the sum of odd number of odd integers is odd, it follows that number of odd vertices in a graph must be even.

Theorem 1.1.3 *A simple graph with n (≥ 2) vertices must have at least two vertices of equal degree.*

Proof : Let G be a simple graph with n(≥ 2) vertices $v_1, \ldots, v_n$. Since G is simple, it has no loop and parallel edges. Hence $d(v_i) \le n - 1$, i = i, ..., n. If possible, let all $d(v_i)$'s be different. Hence the n vertices of G have the following possible degrees : 0, 1, 2, ..., n – 1. Let $d(v_i) = 0$ and $d(v_j) = n - 1$. Then, v_i is an isolated vertex and v_j is adjacent to all other vertices including v_i. So, v_i cannot be isolated. This contradiction proves the theorem.

Corollary 1.1.3 There does not exist a simple graph with n ≥ 2 vertices such that all the vertices are of different degrees.

1.2 Directed Graph

By a directed graph G, briefly called a **digraph**, we mean a triple (V, E, γ) where

(i) V is a non-empty finite set of vertices,

(ii) E is a finite set (may be empty) of directed edges or, arcs and

(iii) $\gamma : E \to V \times V$, is a function that assigns to each arc e an ordered pair (v_i, v_j) of vertices.

We shall use the term **'arc'** for a directed edge. Observe that v_i and v_j may be the same. If $\gamma(e) = (v_i, v_j)$, then v_i is the tail or the starting vertex and v_j is the head or the terminal vertex of the arc e. An arc is said to be directed from its tail to its head. A directed graph is represented pictorially in the same manner as an undirected graph, the only exception being that if $\gamma(e) = (v_i, v_j)$, we put an arrow on the edge e directed from v_i to v_j.

Loop : An arc e of a digraph is called a **loop** if the tail and the head of e are the same.

Example 1.2.1 Let $V = \{v_1, ..., v_4\}$, $E = \{e_1, ..., e_7\}$ and $\gamma(e_1) = (v_1, v_3)$, $\gamma(e_2) = (v_2, v_2)$, $\gamma(e_3) = \gamma(e_4) = (v_1, v_2)$, $\gamma(e_5) = (v_3, v_1)$, $\gamma(e_6) = (v_3, v_4)$, $\gamma(e_7) = (v_2, v_4)$. Then $G = (V, E, \gamma)$ is a digraph which is represented by the following diagram (Fig. 1.2.1).

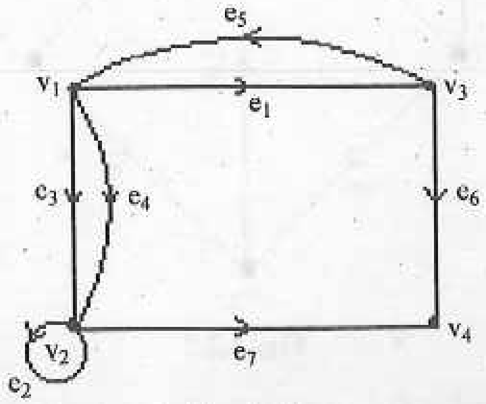


Fig. 1.2.1

In-degree and Out-degree of a vertex in a digraph : Let G be a digraph and v be a vertex of G. The **in-degree** of v is defined as the number of arcs with v as the head, denoted $d^-(v)$. The **out-degree** of v, denoted $d^+(v)$, is the number of arcs with v as the tail.

For example, in the digraph 1.2.1, $d^+(v_1) = 3$, $d^+(v_2) = 2$, $d^+(v_3) = 2$, $d^+(v_4) = 0$ and $d^-(v_1) = 1$, $d^-(v_2) = 3$, $d^-(v_3) = 1$, $d^-(v_4) = 2$. Observe that if e be a loop on the vertex v, then e contributes one in-degree and one out-degree to the vertex v.

Isolated Vertex : A vertex v of a digraph G is called an **isolated vertex** if $d^+(v) = d^-(v) = 0$.

Pendant Vertex : A vertex v of a digraph G is called a **pendant vertex** if it is of degree 1, i.e., if $d^+(v) + d^-(v) = 1$.

Parallel arcs : Two arcs of a digraph are said to be **parallel** if they have the same tail and same head. In the digraph 1.2.1, the arcs e_3, e_4 are parallel whereas e_1 and e_5 are not.

Simple digraph : A digraph that has no loop or parallel arcs is called a **simple digraph**. The following digraph is simple.

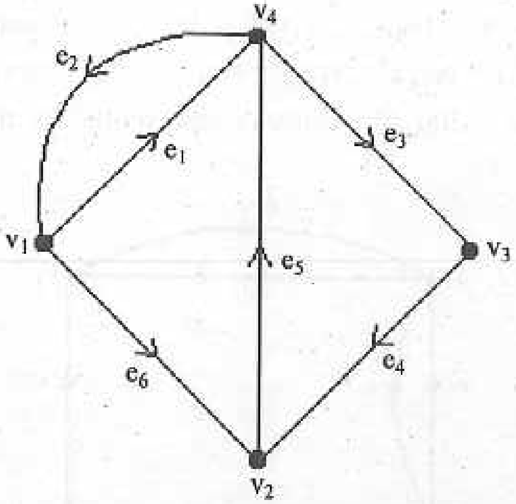


Fig. 1.2.2

Underlying graph : Let G be a digraph. The underlying graph of G is the graph obtained by removing all the edge-directions, i.e., removing all the designations of head and tail from the arcs of G. For example, the graph of fig. 1.2.3 is the underlying graph of the digraph in fig. 1.2.2.

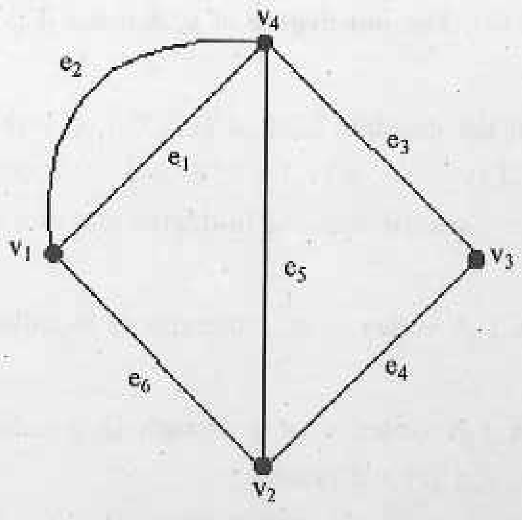


Fig. 1.2.3

Observe that the digraph in fig. 1.2.2 is simple, but its underlying graph in fig. 1.2.3 is not simple.

Theorem 1.2.1 *In any digraph G, the sum of the in-degrees of all the vertices = the sum of their out-degrees = the number of arcs in G.*

Proof : The proof is similar to the proof of theorem 1.1.1. The only difference is that if e is a loop on a vertex v, then e contributes one in-degree and one out-degree to v. Again, if e be an arc with tail v_i and head v_j, then e contributes one in-degree to v_j and one out-degree to v_i. The details of the proof is left as an exercise.

Representation of binary relations on finite sets by digraphs : Let A and B be two non-empty finite sets. A binary relation or simply a relation ρ from A into B is defined as a subset of A × B. If (a,b) ∈ρ, we write a ρ b and say that a is related to b by the relation ρ, and if (a,b) ∉ρ, we say that a is not related to b by ρ and express this fact by a $\not\rho$ b. If A = B, then we simply say that ρ is a relation on A.

Throughout this section we shall consider only finite sets.

We now show how a binary relation on a finite set can be represented by a digraph. Let ρ be a relation on a finite set $A=\{a_1, \ldots\ldots, a_n\}$. We represent each a_i by a dot or a small circle as a vertex of a digraph. If a_i ρ a_j holds, we draw an arc from the vertex a_i to the vertex a_j. We do this for all the related pairs in A. Clearly, the resulting pictorial representation of ρ is a digraph without parallel arcs. The converse is also true, i.e. every digraph without parallel arcs defines a relation on the set of its vertices.

Example 1.2.2 Let A = {2, 3, 5, 7, 9} and ρ be the relation on A defined as follows : for a, b ε A, a ρ b holds if and only if a < b.

We now construct a digraph of the relation ρ.

We represent the five numbers 2, 3, 5, 7, 9 by five dots or small circles and treat them as vertices of a graph. We observe that ρ = {(2, 3), (2, 5), (2, 7), (2, 9), (3, 5), (3, 7), (3, 9), (5, 7), (5, 9), (7, 9)}.

Since (2, 3) ε ρ, we draw an arc from 2 to 3. Similarly, we draw arcs for other pairs in ρ. The resulting digraph, as shown in fig. 1.2.4, represents the relation ρ.

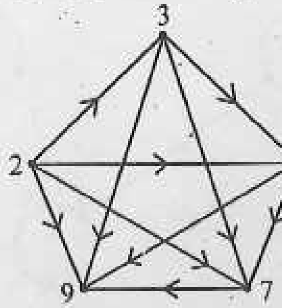


Fig. 1.2.4

We can also find the relation on the set of vertices of a digraph without parallel arcs.

Example 1.2.3. We consider the following digraph as shown in fig. 1.2.5.

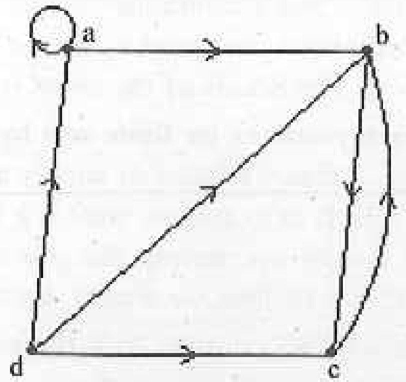


Fig. 1.2.5

This digraph defines a relation ρ on the set A = {a,b,c,d}. From the arcs in the digraph we see that the relation ρ is defined as :

ρ = {(a,a), (a,b), (b,c), (c,b), (d,a), (d,b), (d,c)}

Reflexive digraph : A relation ρ on a set A is called **reflexive** if a ρ a holds for all a ε A. Clearly, the corresponding digraph representing ρ will have a loop at every vertex. This digraph is called a reflexive digraph.

Example 1.2.4. Let A = {3, 4, 5, 6, 7} and ρ be the relation on A defined as follows : for a, b ε A, a ρ b holds if and only if a divides b. Then. ρ = {(3,3), (3,6), (4,4), (5.5). (6.6), (7,7)}. The relation ρ is reflexive and its corresponding reflexive digraph is the following : (Fig. 1.2.6)

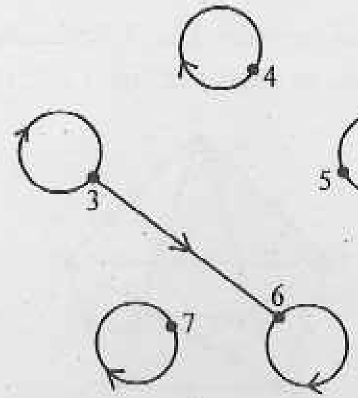


Fig. 1.2.6

Symmetric digraph : A relation ρ on a set A is called **symmetric** if a ρ b implies b ρ a for all a, b ε A. Clearly, the corresponding digraph G representing ρ will be such that for every arc (a,b) there is also an arc (b,a) where a, b are the vertices of G. Such a digraph is called a symmetric digraph.

Example 1.2.5. Let A = $\{l_1, l_2, l_3, l_4\}$ be a set of straight lines and ρ be the relation on A defined as follows : for $l_i, l_j \in A$, $l_i \rho l_j$ holds if l_i is perpendicular to l_j. It is given that $l_1 \perp l_2$, $l_1 \perp l_4$, $l_2 \perp l_3$. This relation is symmetric and its corresponding symmetric digraph is the following : (Fig. 1.2.7)

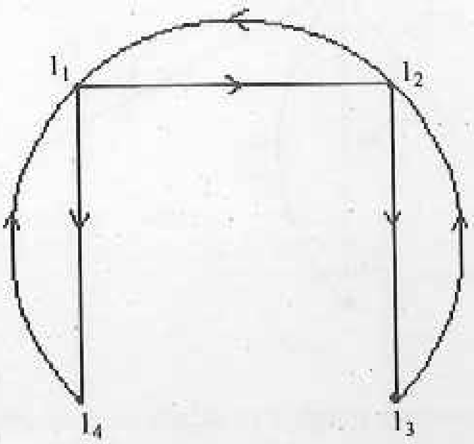


Fig. 1.2.7

Transitive digraph : A relation ρ on a set A is called **transitive** if for any three elements a, b, c ε A, a ρ b and b ρ c together imply a ρ c. The digraph which represents a transitive relation is called a transitive digraph. For example, the digraph as shown in the fig. 1.2.4 for the example 1.2.2 is a transitive digraph.

1.3 Worked Out Exercises

1. Draw a graph having the given properties or, explain why no such graph can exist.

(a) Simple graph with degree-sequence (2,2,4,4,4)

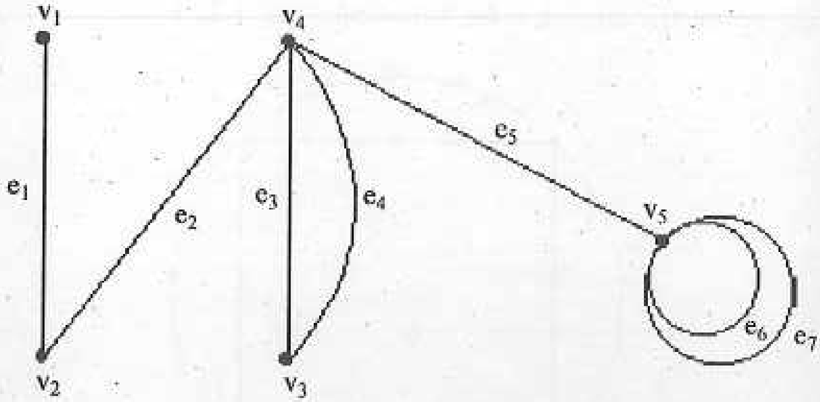
(b) Four edges and degree-sequence (2,2,3,3)

(c) Degree-sequence (1,2,2,4,5).

Solution : (a) If possible, let there be a simple graph G with five vertices $v_1, \ldots, v_5$ having respective degrees 2,2,4,4,4. Since G has no loop and no parallel edges, it follows that each of the vertices v_3, v_4, v_5 (say) must have four adjacent vertices. Hence v_1 (also v_2) must be an adjacent vertex of v_3, v_4, v_5. Consequently, $d(v_1)$ must be at least 3, which contradicts the given condition. Hence there cannot exist any such graph.

(b) Sum of the degrees of the vertices = 2+2+3+3=10. So. there must be five edges, a contradiction. Hence no such graph can exist.

(c)



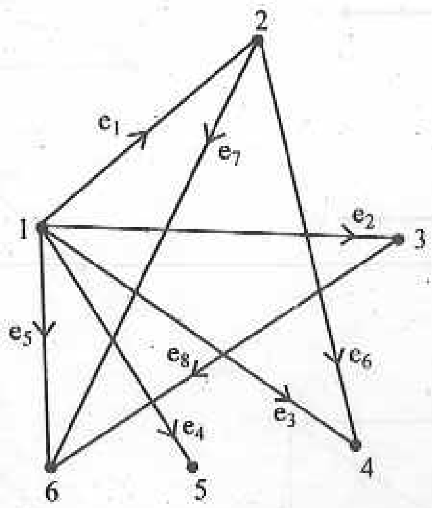
2. Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Solution : Let G be a simple graph with n vertices. Let v_i be an arbitrary vertex of G. Since G is simple, the maximum number of edges incident with v_i is $n-1$. Hence, max $d(v_i) = (n-1)$. This is true for each of the vertices. Hence, $\sum_{i=1}^{n} \max d(v_i) = n(n-1)$. So, twice the maximum number of edges = $n(n-1)$. Hence, maximum number of edges in G = $\frac{n(n-1)}{2}$.

3. Let $G = (V, E, \gamma)$ be a digraph, where $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{e_1, \ldots\ldots, e_n\}$ and $\gamma(E) = \{(x,y) : x, y \in V, x \neq y$ and x divides $y\}$. Find n. Draw this digraph. Is it a simple digraph?

Solution : 1 divides the remaining five numbers 2,3,4,5,6. Hence the number of arcs with tail 1 is five. Similarly, 2 divides 4 and 6. So, the number of arcs with tail 2 is two. 3 divides only 6. Hence there is only one arc with tail 3. Thus the total number of arcs in the digraph = n = 5 + 2 + 1 = 8.

We draw the digraph as follows :

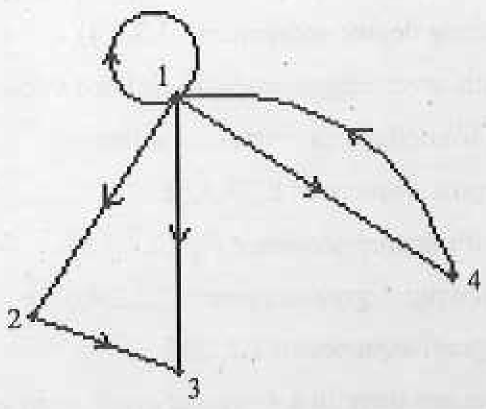


Since there are no parallel arcs and no loops, the digraph is simple.

4. Draw the digraph for the relation ρ on the set S = {1, 2, 3, 4} given by ρ = {(1,1), (1,2), (1,4), (4,1), (2,3), (1,3)}.

Solution : We label the four vertices of the digraph as 1,2,3,4. We then join the vertices by arcs according to the given relation.

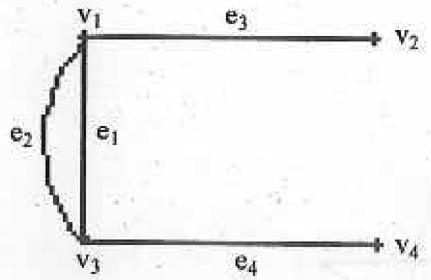
The digraph is :



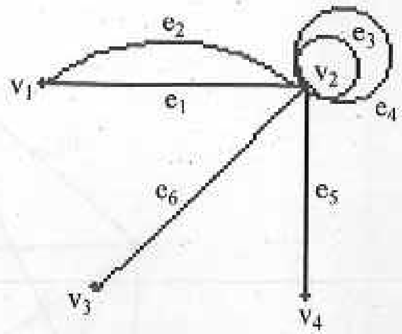
1.4 Exercises

1. State which of the following graphs are simple. Find also the degree of each vertex of the given graphs :

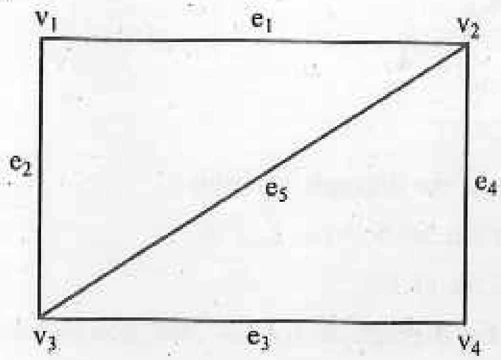
(a)



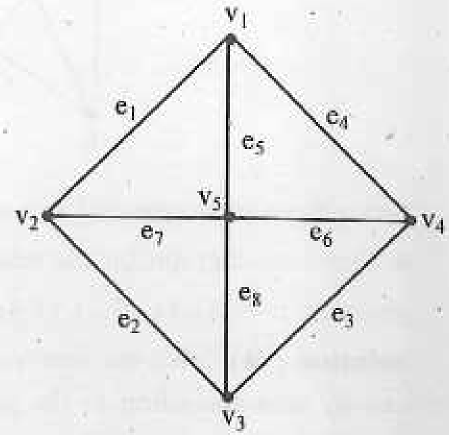
(b)



(c)



(d)



2. Draw a graph having the given properties or, explain why no such graph can exist.

(a) Simple graph having degree-sequence (3,3,3,3,4)

(b) Simple graph with seven edges, nine non-isolated vertices.

(c) Six vertices and four edges, no vertex is isolated.

(d) A graph with degree-sequence (1,2,3,4,5).

(e) Six edges and with degree-sequence (1,1,2,4,5,5).

(f) A simple graph having degree-sequenec (2,2,2,4,5,5).

(g) A graph with degree-sequence (0,1,2,2,3).

3. How many vertices are there in a 4-regular graph with ten edges?

4. Can there be a simple graph with seven vertices and twenty four edges?

5. Can there be a regular graph with eight vertices and seven edges?

6. Draw two different graphs with the same degree-sequence (3.3.3.3).

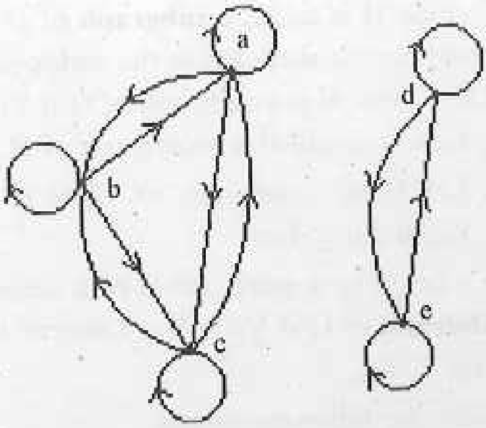
7. In a group of seven people, is it possible for each person to shake hands with exactly three other people? Justify your answer. (*Hints :* The resulting graph will have the sum of the degrees of the vertices (Persons) = 7 × 3 = 21).

8. In a group of n (>1) people, is it true that there are at least two people with exactly the same number of friends? Justify your answer.

9. Draw a digraph with the following in-degree and out-degree sequences : in-degree : (1,1,1,1), out-degree : (1,1,1,1).

10. Let A = {a, b, c, d} and ρ be the relation on A defined as ρ = {(a,a), (a,b), (b,a), (a,c), (c,a), (b,b), (c,c), (d,d), (b,d), (d,b)}. Draw the digraph representing ρ. Is it reflexive? Is it symmetric?

11. Find the relation represented by the following digraph. Is the relation (i) reflexive, (ii) symmetric, (iii) transitive?



Unit 2 □ Subgraphs, Isomorphism of graphs, walks, paths, cycles

Structure

2.1 Subgraphs

2.2 Isomorphism of graphs

2.3 Walks, Paths, Cycles

2.4 Exercises

2.1 Subgraphs

Let G be a graph. A graph H is called a **subgraph** of G if all the vertices and edges of H are in G. Throughout we shall denote the vertex-set of a graph G by V_G and the edge-set of G by E_G. Thus, H is a subgraph of G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. If H is a subgraph of G, then G is called a supergraph of H.

Proper Subgraph : Let H be a subgraph of G. H is said to be a **proper subgraph** of G if $V_H \subset V_G$ or $E_H \subset E_G$.

Spanning Subgraph : Let G be a graph and H be a subgraph of G. Then, H is said to be a **spanning subgraph** of G if $V_H = V_G$. Observe that if the subgraph H spans the graph G, then $E_H \subseteq E_G$.

Example 2.1.1 Consider the following graphs :

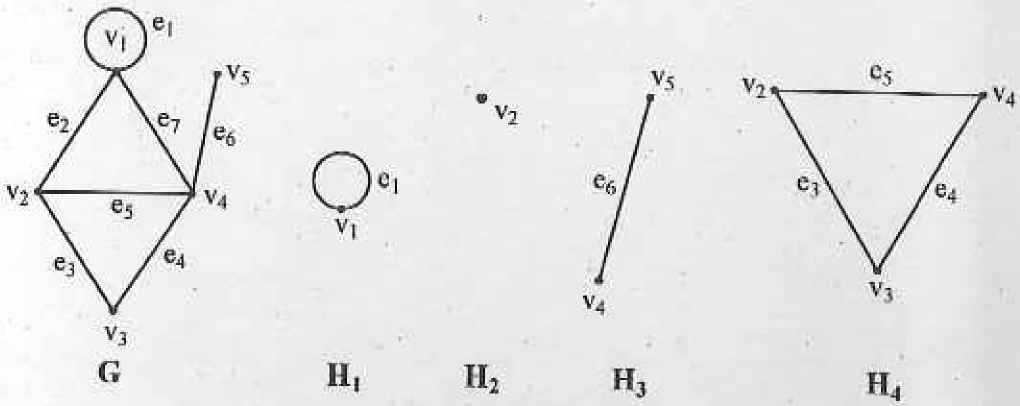


Fig. 2.1.1

The graphs H_1, H_2, H_3, H_4 are subgraphs of G.

Observe the following properties of a subgraph.

(i) Every graph is its own subgraph.

(ii) A single vertex of a graph G forms a subgraph of G.

(iii) A single edge with its end vertices of a graph G forms a subgraph of G.

Example 2.1.2 Consider the following graphs :

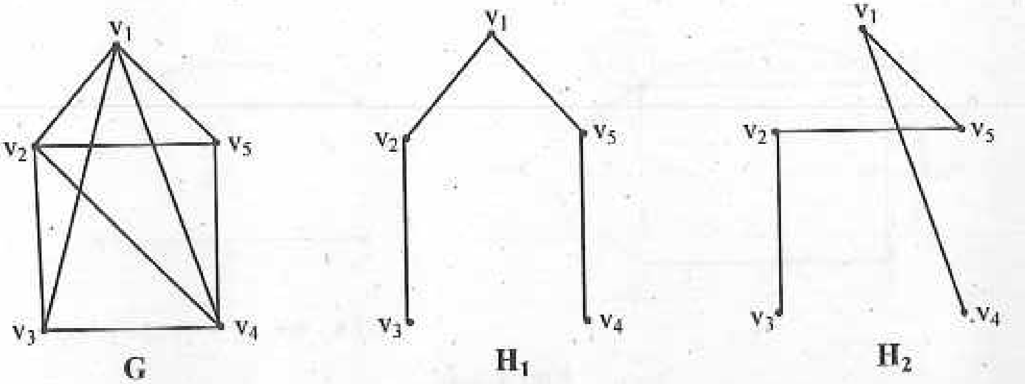


Fig. 2.1.2

The graphs H_1 and H_2 are subgraphs of the graph G. Moreover, $V_{H_1} = V_{H_2} = V_G$. Hence both H_1 and H_2 are spanning subgraphs of G.

Induced subgraph : Let G be a graph and $S \subseteq V_G$.

Then the subgraph of G whose vertex-set is S and whose edge-set consists of all the edges of G having both end points in S is said to be the **subgraph induced** by S. We denote this subgraph by G(S). It is the maximal subgraph of G with vertex-set S. Two vertices of S are adjacent in G(S) if and only if they are adjacent in G.

Example 2.1.3 Consider the following graphs :

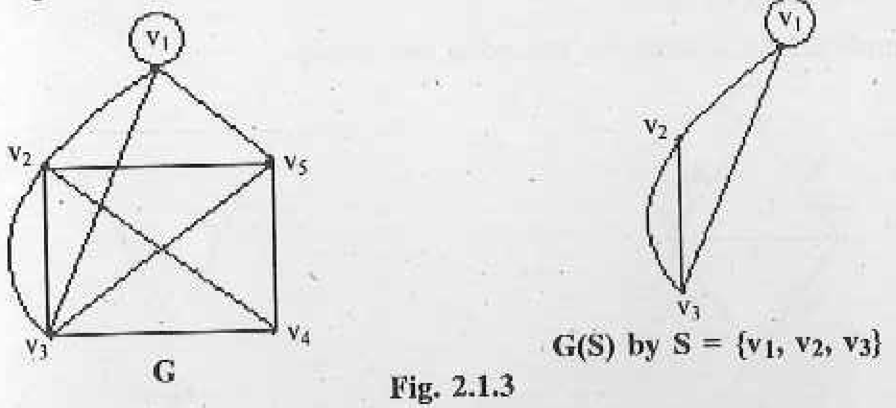
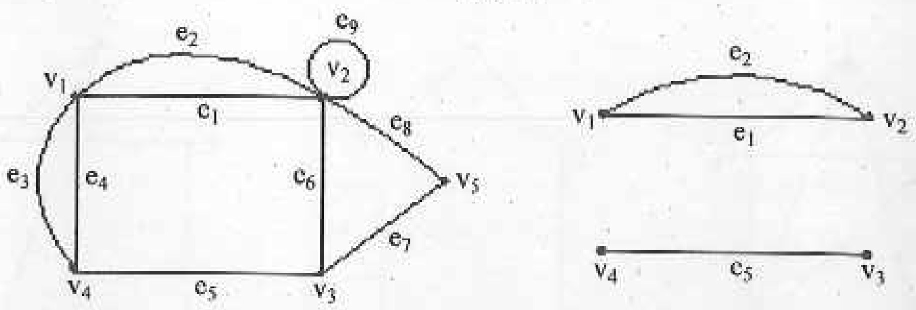


Fig. 2.1.3

The subgraph G(S) is the induced subgraph of G by the vertex-set $S = \{v_1, v_2, v_3\}$.

Induced subgraph by an edge-set : Let G be a graph and $A \subseteq E_G$. Then the subgraph of G whose edge-set is A and whose vertex-set consists of all the vertices of G that are incident with at least one edge in A is said to be the **subgraph induced** by the edge-subset A. We denote this subgraph by G(A).

Example 2.1.4 Consider the following graphs :



G **G(A) by A = {e_1, e_2, e_5}**

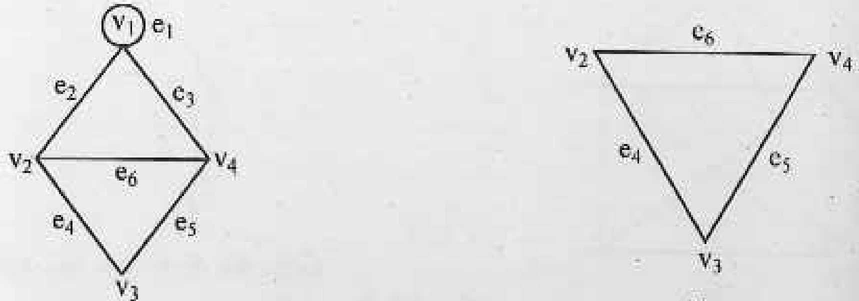
Fig. 2.1.4

The subgraph G(A) is the subgraph of G induced by the edge-subset A = {e_1, e_2, e_5}.

Deletion of a vertex from a graph : Let G be a graph and v be a vertex of G. Then G–v is the subgraph of G obtained by deleting the vertex v and also from G deleting all the edges incident with v. Thus, G–v is the subgraph induced by the vertex-set $V_G-\{v\}$. It is the maximal subgraph of G not containing v.

We may generalize the deletion operation for a set of vertices $S \subset V_G$ by deleting all the vertices of S and associated edges successively from G. The resulting subgraph is denoted by G–S.

Example 2.1.5 Consider the following two graphs :



G **G–v_1**

Fig. 2.1.5

The subgraph $G-v_1$ is obtained from G by deleting the vertex v_1 and the edges e_1, e_2, e_3 incident on v_1 from G.

Deletion of an edge from a graph : Let G be a graph and e be an edge of G. Then G–e is the subgraph of G obtained by deleting only the edge e from G. Thus, $V_{G-e} = V_G$ and $E_{G-e} = E_G - \{e\}$. G–e is the maximal subgraph of G not containing e.

We may generalize the deletion operation for a set of edges $A \subseteq E_G$ by deleting all the edges in A successively from G. The resulting subgraph is denoted by G–A.

Note : Deletion of a vertex v from a graph means deletion of the vertex v as well as the deletion of edges incident with v, but deletion of an edge from a graph does not delete its end vertices.

Example 2.1.6 Consider the following two graphs :

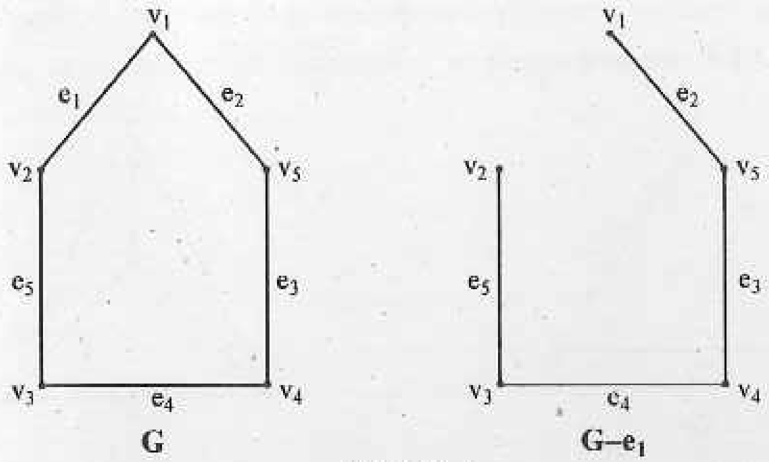


Fig. 2.1.6

The subgraph $G-e_1$ is obtained from G by deleting the edge e_1 from G.

Addition of a vertex to a graph : Let G be a graph and v be a new vertex not in V_G. Then the supergraph, denoted $G \cup \{v\}$, is obtained by adding the new vertex v to the graph G. This supergraph has the vertex-set $V_G \cup \{v\}$ and edge-set E_G.

Addition of an edge to a graph : Let G be a graph and u, v be two vertices of G. If we add a new edge $e \notin E_G$ joining u and v, then the resulting supergraph, denoted $G \cup \{e\}$, is the graph whose vertex-set is V_G and edge-set is $E_G \cup \{e\}$.

Example 2.1.7 The following three graphs illustrate the addition operations.

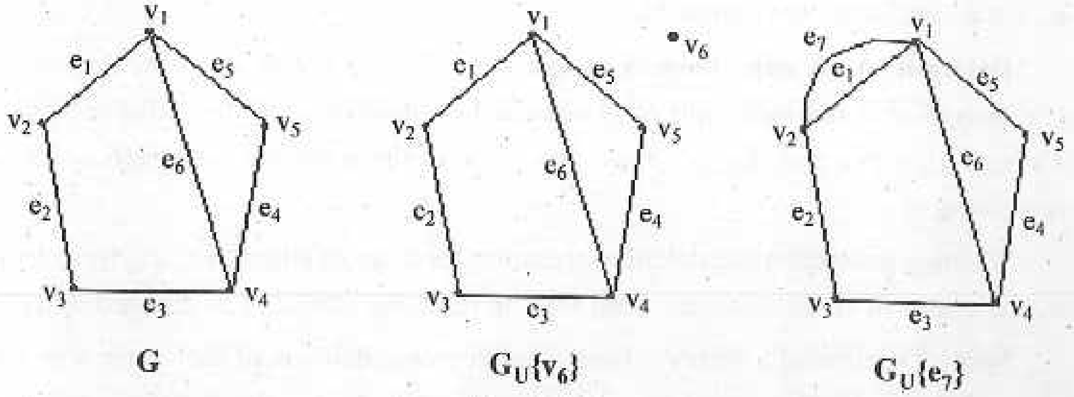


Fig. 2.1.7

Join of a graph and a vertex : Let G be a graph and v be a new vertex not in V_G. If we add the vertex v to G and then join each of the vertices of V_G with v by a new edge, the resulting graph is called the join of G and v and is denoted by G+v.

Example 2.1.8 The join operation is illustrated by the following graphs :

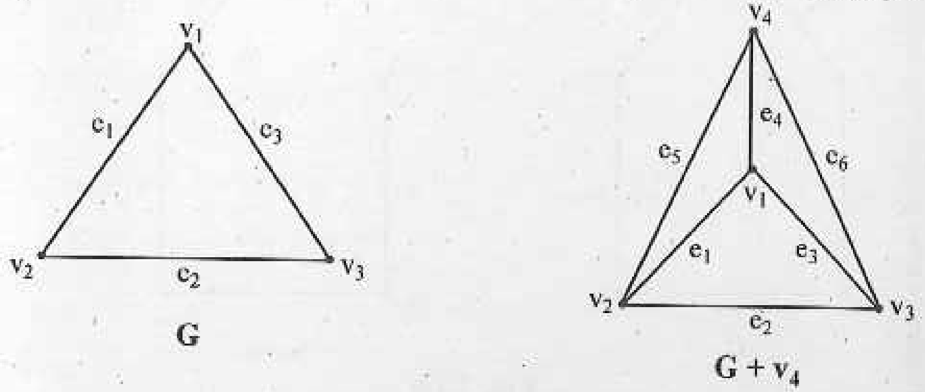


Fig. 2.1.8

2.2 Isomorphism of graphs

Two graphs G and H are said to be **isomorphic** if there exist a one-to-one correspondence $f : V_G \to V_H$ and also a one-to-one correspondence $\phi : E_G \to E_H$ such that for every $e \in E_G$ with end points u, v in V_G, the function f maps u and v to the end points of $\phi(e)$ in V_H.

We write $G \cong H$ or, $G = H$ to indicate the fact that G and H are isomorphic. We then say that G and H are the same graphs. Otherwise, we call them different.

Example 2.2.1 Consider the following two graphs :

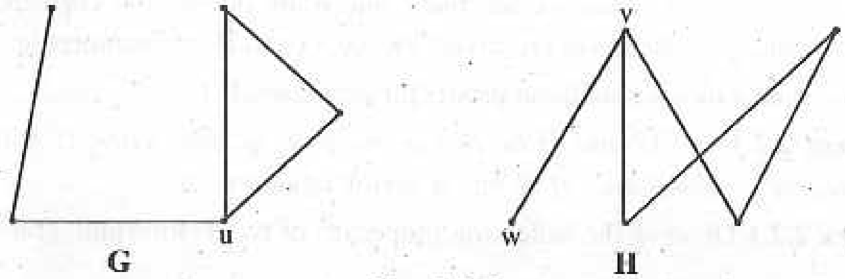


Fig. 2.2.1

Both the graphs G and H have five vertices, five edges and degree-sequence (1,2,2,2,3). But the two graphs are not isomorphic. Note that G has a triangle (a cycle of length 3) while H has no triangle (see the next section for the definition of a cycle).

Definition 2.2.1 Let G and H be two graphs. A one-to-one correspondence f : $V_G \rightarrow V_H$ is said to **preserve adjacency** if for all u, v ε V_G, f(u) is adjacent to f(v) in H if and only if u is adjacent to v in G. We now state a few properties of isomorphic graphs. The proofs can be found in any standard book on graph theory.

Let G and H be two isomorphic graphs. Then,

(i) A loop in G must be mapped to a loop in H.

(ii) Two simple graphs G and H are isomorphic if and only if there exists a one-to-one correspondence f : $V_G \rightarrow V_H$ such that f preserves adjacency.

(iii) The relation $\cong$ (isomorphic to) defined in the set of graphs is an equivalence relation.

(iv) Let G and H be two isomorphic graphs and f : $V_G \rightarrow V_H$ be a one-to-one correspondence for the graph isomorphism. Then any vertex u in V_G and f(u) have the same degree.

Example 2.2.2 Consider the following two graphs :

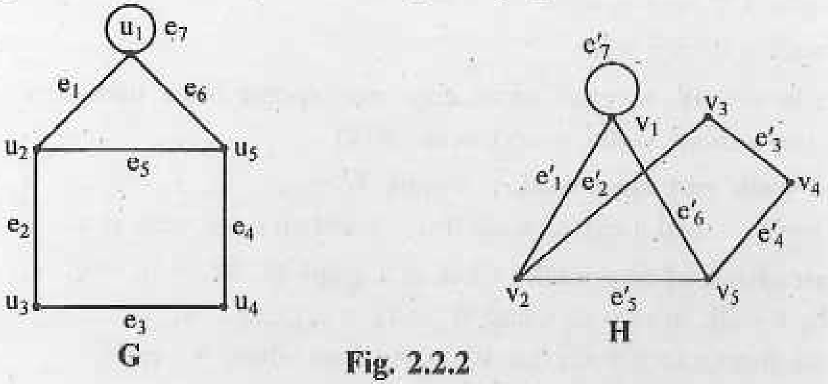


Fig. 2.2.2

We define $f : V_G \to V_H$ by $f(u_i) = v_i$, $i = 1, \ldots\ldots, 5$ and $\phi : E_G \to E_H$ by $\phi(e_j) = e'_j$, $j = 1, \ldots\ldots, 7$. Then we see that f and ϕ are one-to-one correspondences such that the incidence relation is preserved. Hence, G and H are isomorphic.

We now state a theorem without proof (for proof, see [1]).

Theorem 2.2.1 *Let G and H be two isomorphic graphs. Then G will have a vertex of degree p if and only if H has a vertex of degree p.*

Remark 2.2.1 Observe the following properties of two isomorphic graphs G and H.

(i) $n(V_G) = n(V_H)$, where $n(V_G)$ denotes the number of vertices in G.

(ii) $n(E_G) = n(E_H)$, where $n(E_G)$ denotes the number of edges in G.

(iii) Degree-sequence of G = degree-sequence of H.

But the converse is not true. That is, two graphs may satisfy all the above three conditions, yet they may not be isomorphic. We have shown this in the example 2.2.1.

2.3 Walks, Paths, Cycles

We now define the following terms for a graph.

Walk : Let G be a graph and $u, v \in V_G$. A **walk** in G from u to v is an alternating sequence $W = (u = v_1, e_1, v_2, e_2, \ldots\ldots, v_{n-1}, e_{n-1}, v_n, e_n, v_{n+1} = v)$ of vertices and edges of G such that endpoints of e_i are the vertices v_i and v_{i+1}, for $i = 1, \ldots\ldots, n$.

Length of a walk : The **length of a walk** is the number of occurrences of edges in the walk sequence.

Subwalk : Let $W = (v_1, e_1, v_2, \ldots\ldots, e_n, v_{n+1})$ be a walk in a graph G. A **subwalk** W_1 of the walk W is a subsequence of consecutive entries $W_1 = (v_i, e_i, v_{i+1}, \ldots\ldots, e_k, v_{k+1})$ such that $1 \le i \le k \le n+1$.

A subwalk is itself a walk.

Note : In a walk, a vertex or an edge may appear more than once. A walk of length o is just a single vertex v(say), denoted (v).

Closed walk and open walk : A walk $W = (u = v_1, e_1, v_2, \ldots\ldots, v_n, e_n, v_{n+1} = v)$ from u to v is called a closed walk if u = v and an open walk if $u \neq v$.

Concatenation of two walks : Let in a graph G, $W_1 = (u = v_1, e_1, v_2, \ldots\ldots, e_k, v_{k+1} = v)$ be a walk from u to v and $W_2 = (v = v_{k+1}, e_{k+1}, v_{k+2}, \ldots\ldots, e_n, v_{n+1} = w)$ be another walk from v to w such that W_2 starts from where W_1 ends.

Then the concatenation of two walks W_1 and W_2, denoted $W_1.W_2$ is defined as the walk $W_1.W_2 = (u = v_1, e_1, v_2, \ldots\ldots, e_k, v_{k+1} (= v), e_{k+1}, v_{k+2}, \ldots\ldots, e_n, v_{n+1} = w)$

Example 2.3.1 Consider the following graph G :

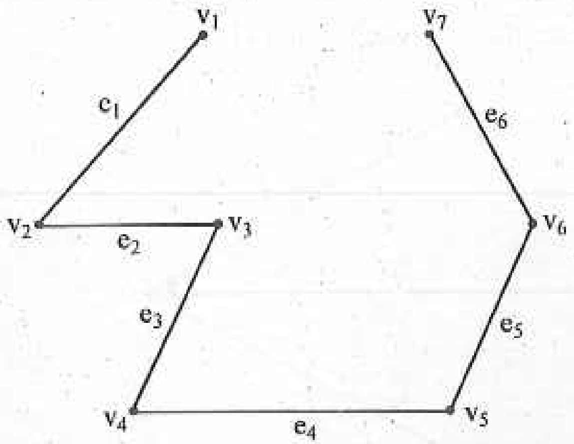


Fig. 2.3.1

In G, let $W_1 = (v_1, e_1, v_2, e_2, v_3)$ and $W_2 = (v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, v_7)$.

Then concatenation of W_1 and W_2 is the walk $W_1.W_2 = (v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6, e_6, v_7)$.

Trail : A walk with no repeated edges i.e. all the edges in the walk are different is called a trail.

The trail is called closed if the starting and terminal vertices of the trail coincide.

Path : A path is a trail with no repeated vertices, except possibly the starting and terminal vertices.

If starting and terminal vertices of a path coincide, then it is called a closed path.

The length of a path is the number of occurrences of edges in the path.

Trivial walk (or, trail, or, path) : A walk or a trail or a path is said to be trivial if it has only one vertex and no edge. Otherwise, it is non-trivial.

Circuit : A non-trivial closed trail is called a circuit. Every loop is a circuit. A circuit must have non-zero length.

Cycle : A non-trivial closed path is called a cycle.

A cycle is always a circuit, but the converse is not true.

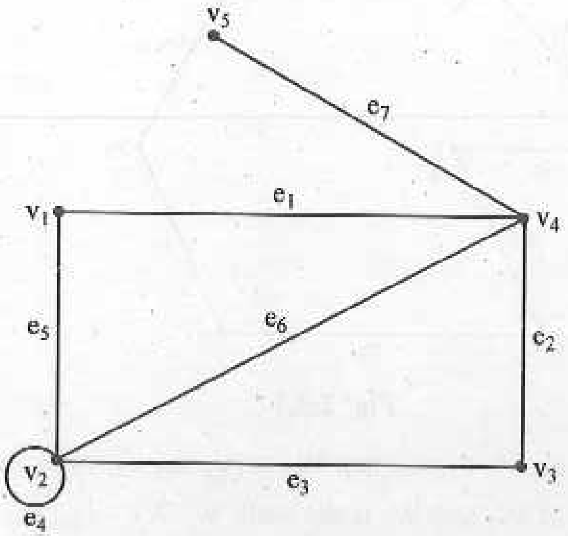
Even and Odd cycles : A cycle is called even (odd) if its length is even (odd).

A cycle with three edges is called a triangle.

Distance between two vertices : Let G be a graph and u, v ε V_G. The distance between u and v, denoted d(u, v), is the length of a shortest walk, if there exists any, from u to v.

If there does not exist any walk from u to v, we write d(u, v) = ∞.

Example 2.3.2 Consider the following graph G :



G

Fig. 2.3.2

In the graph 2.3.2 we consider the following walks :

$W_1 = (v_1, e_1, v_4, e_7, v_5)$ is a walk from v_1 to v_5. It is an open walk with length two. Again, $W_2 = (v_1, e_5, v_2, e_4, v_2, e_6, v_4, e_1, v_1)$ is a closed walk from v_1 to v_1 whose length is 4.

$W_3 = (v_1, e_5, v_2, e_3, v_3, e_2, v_4, e_6, v_2)$ is an open trail and

$W_4 = (v_4, e_1, v_1, e_5, v_2, e_3, v_3, e_2, v_4)$ is a closed trail. It is also a closed path.

$W_5 = (v_2, e_3, v_3, e_2, v_4)$ is a path.

$W_6 = (v_2, e_4, v_2)$ is a circuit. W_4 is also a circuit and a cycle of even length. The walk W_2 is a circuit, but not a cycle.

$W_7 = (v_1, e_5, v_2, e_6, v_4, e_1, v_1)$ is a triangle.

Theorem 2.3.1 *If the degree of each vertex of a graph G is at least two, then G contains a circuit.*

Proof : Let G be a graph such that the degree of each vertex of G is at least two. If G consists of a single vertex v_1, then there must be at least one loop e_1 at v_1 since $d(v_1) \geq 2$. In that case, (v_1, e_1, v_1) is a circuit. We now assume that G contains more than one vertices without any loop. We start from an arbitrary vertex v_1 and follow along the edges. Since $d(v_1) \geq 2$, there are at least two edges incident with v_1. Let one of these edges be between v_1 and v_2. We arrive at v_2 along this edge. Since $d(v_2) \geq 2$, v_2 must have at least another edge, say, between v_2 and v_3. We can arrive at v_3 along this edge. Therefore, if we arrive at a vertex where we were not previously, then we can continue. Proceeding in this way, we ultimately arrive at a vertex that has previously been traversed, thus forming a circuit in G.

Theorem 2.3.2 *Let G be a graph and u, v be two distinct vertices of G. If there is a walk from u to v, then there is a path from u to v.*

Proof : Let $W = (u = v_1, e_1, v_2, \ldots\ldots, e_{n-1}, v_n = v)$ be a walk from u to v in G.

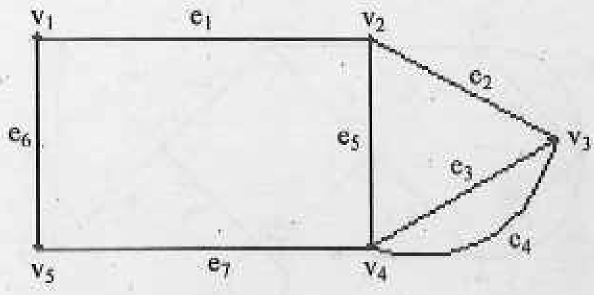
If this is not a path, then $v_i = v_j$ for some i, $j(1 \leq i < j \leq n)$. Hence there is a closed walk from v_i to v_i $(= v_j)$ in W. We delete this closed walk (excluding one v_i) from W and form the new walk W_1 from u to v. If W_1 is not a path, we repeat the above process and ultimately we obtain a path from u to v.

Theorem 2.3.3 *Every circuit in a graph contains a cycle.*

Proof : Let C be a circuit in a graph G. Among all the non-trivial closed subwalks of C, let T be a subwalk having minimum length. Since the length of T is minimum, it has no proper closed subwalks. This implies its only repeated vertices are the starting vertex and the terminal vertex. Thus, T is a cycle.

2.4 Exercises

1. Determine whether the given walk in the following graph is (a) a trail, (b) a path, (c) a closed walk, (d) a circuit, (e) a cycle.



(i) $W_1 = (v_1, e_1, v_2)$

(ii) $W_2 = (v_2, e_2, v_3, e_3, v_4, e_4, v_3)$

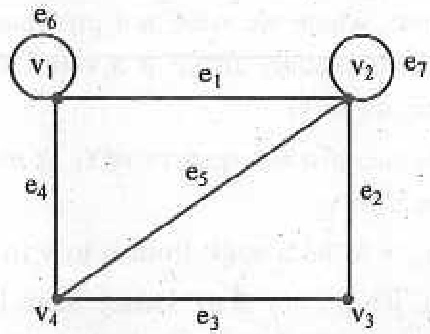
(iii) $W_3 = (v_2, e_2, v_3, e_3, v_4, e_4, v_3, e_2, v_2)$

(iv) $W_4 = (v_4, e_7, v_5, e_6, v_1, e_1, v_2, e_2, v_3, e_3, v_4)$

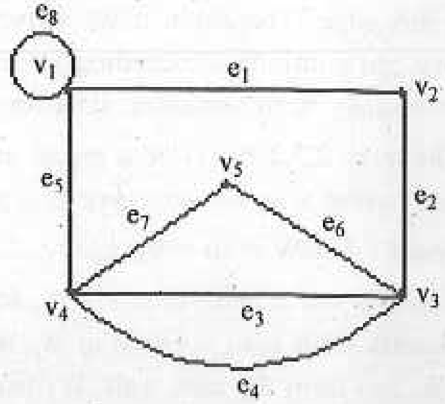
(v) $W_5 = (v_4, e_4, v_3, e_3, v_4, e_5, v_2, e_1, v_1, e_6, v_5, e_7, v_4)$

2. Find three proper subgraphs of each of the following graphs :

(a)

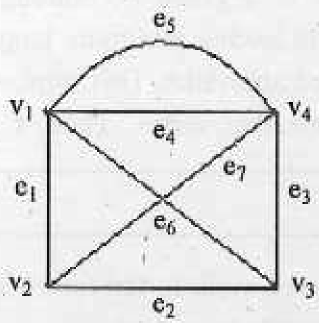


(b)

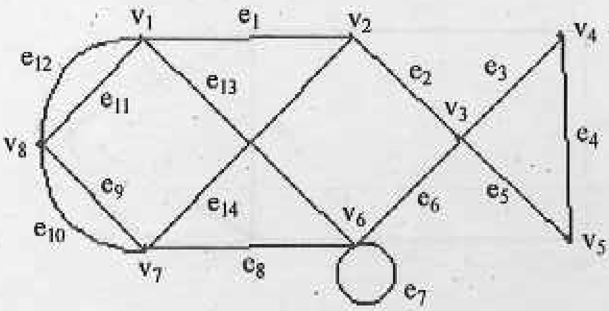


Find also a spanning subgraph in each of (a) and (b).

3. Find two cycles of different lengths in the following graph. Find also the length of each of the cycles.

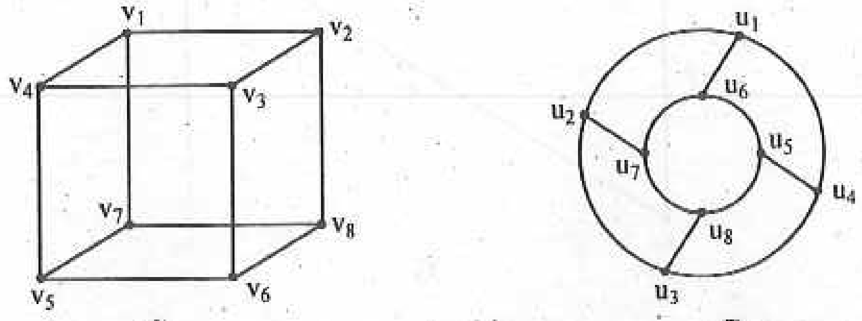


4. Let G be the following graph :



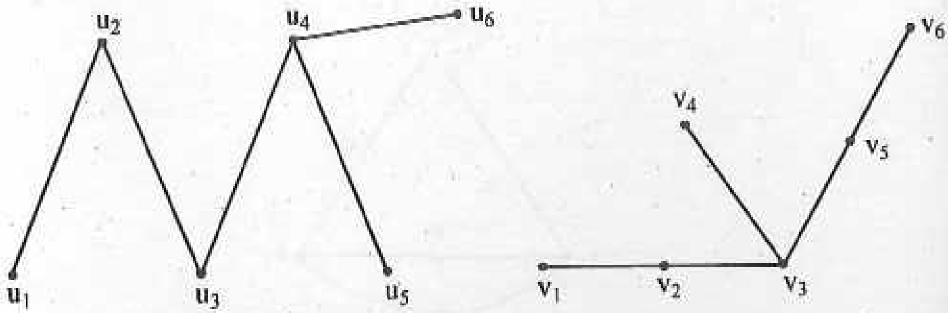
In G, find (a) a walk of length 8 from v_1 to v_6, (b) a trail of length 7 from v_6 to v_4, (c) a circuit of length 7 from v_7 to v_7, (d) a cycle of length 6 from v_3 to v_3. Does G contain an odd cycle from v_2 to v_2? Find the distances between v_1 and v_5 and between v_4 and v_8.

5. Show that the following two graphs G_1 and G_2 are isomorphic.



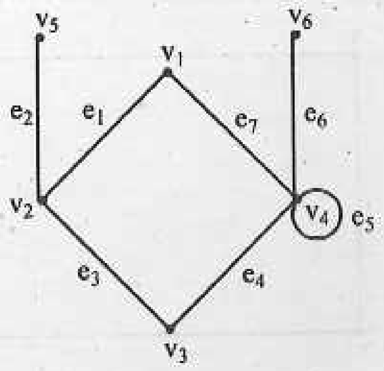
G_1 G_2

6. Show that the following two graphs G_1 and G_2 are not isomorphic.



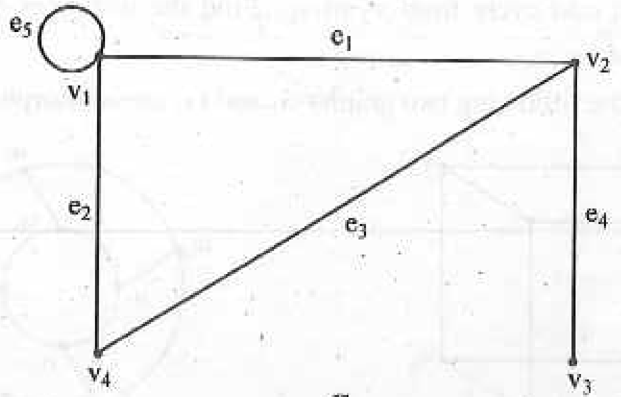
G_1 G_2

7. For the following graph G, find the induced subgraph G(S) by the vertex-set $S = \{v_1, v_2, v_4\}$.



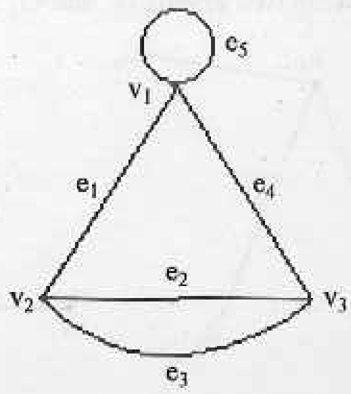
G

8. For the following graph G, find the induced subgraph G(A) of G on the edge-set $A = \{e_1, e_2\}$.



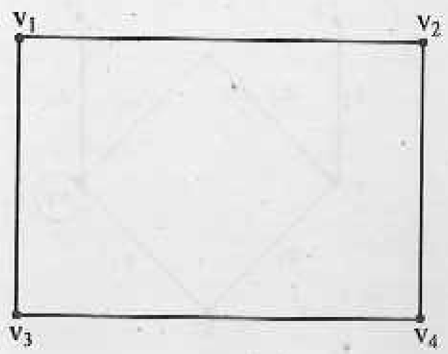
G

9. For the following graph G, find $G - v_1$, $G - e_1$ where v_1 is a vertex of G and e_1 is an edge of G.



G

10. For the following graph G, find the join G+v, where $v \notin V_G$.



G

Unit 3 □ Connected graphs, complement of a graph, bipartite graphs

Structure

3.1 Connected graph

3.2 Complete graph

3.3 Bipartite graphs

3.4 Worked Out Exercises

3.5 Exercises

3.1 Connected graph

Let G be a graph and u, v ε V_G. We say that the vertex v is **reachable** from the vertex u if there is a walk from u to v.

A graph G is said to be connected if for every pair of vertices u and v in G, there is a walk from u to v. In otherwords, G is connected if any vertex in G is reachable from any other vertex. Otherwise, G is said to be disconnected.

A null graph with one vertex is connected whereas a null graph with more than one vertices is disconnected.

Example 3.1.1 Consider the following graphs G_1, G_2, G_3.

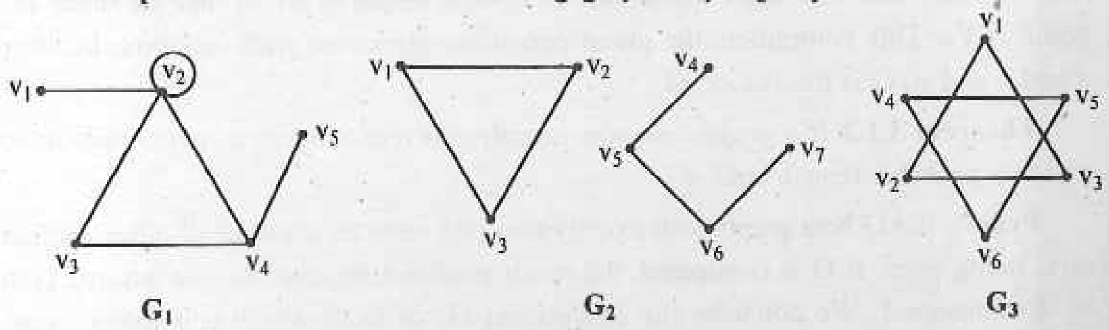


Fig. 3.1.1

The graph G_1 is connected whereas the graphs G_2 and G_3 are disconnected.

Component of a graph : A subgraph H of a graph G is called a **component** of G if H is connected and is not properly contained in any connected subgraph of G.

Let $G = (V, E, \gamma)$ be a graph. We define a relation ρ, called reachability relation, on V by $\rho = \{(u, v) \varepsilon V \times V \mid$ there exists a walk from u to v in G$\}$. Evidently, ρ is an equivalence relation on V.

Let V_1 be an equivalence class of ρ and E_1 be the set of edges joining the vertices in V_1. We take $\gamma_1 = \gamma/E_1$. Then, $G_1 = (V_1, E_1, \gamma_1)$ is a subgraph of G which is connected and is not properly contained in any connected subgraph of G. Thus, G_1 is a component of G. In fact, every equivalence class of ρ is a component of G. Hence, "every graph can be decomposed into finite number of components". Consequently, "A graph G is connected if and only if G has only one component".

Theorem 3.1.1 *Let G be a graph whose vertex-set is V. G is disconnected if and only if there exist two non-empty subsets V_1 and V_2 such that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \phi$ and there is no edge in G whose one end point is in V_1 and the other end point is in V_2.*

Proof : Let G be disconnected and $u \varepsilon V$. Let $V_1 = \{v \varepsilon V \mid$ there is a path from u to v in G$\}$ and $V_2 = V - V_1$. Then, $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$. Also $V_1 \neq \phi$ since $u \varepsilon V_1$. Since G is disconnected, $V_1 \neq V$ and hence $V_2 \neq \phi$. If possible, suppose there is an edge e in G whose one end point $u_1 \varepsilon V_1$ and the other end point $u_2 \varepsilon V_2$. Since u_1 is joined to u by a path p (say), u_2 is also joined to u by a path (p, e, u_2). Hence $u_2 \varepsilon V_1$, a contradiction. Hence no vertex in V_1 is joined to any in V_2 by an edge. The subsets V_1 and V_2 form the required partition of V.

Conversely, suppose that such a partition of V exists. We take two vertices u and v of G such that $u \varepsilon V_1$ and $v \varepsilon V_2$. If there exists a path between u and v, then there must have at least one edge whose one end point would be in V_1 and the other end point in V_2. This contradicts the given condition. Hence no path can exist between u and v and so G is disconnected.

Theorem 3.1.2 *If a graph contains exactly two odd vertices u and v, then there exists a path between u and v.*

Proof : Let G be a graph with exactly two odd vertices u and v, all other vertices of G being even. If G is connected, the result is obviously true. So, we assume G to be disconnected. We consider the component G_1 of G to which u belongs. Now, G_1 is a connected subgraph of G. It is known that (cf. theorem 1.1.2) number of odd-degree vertices in any graph is even. Since u and v are the only odd-degree vertices in G, v must also belong to G_1. Hence u and v must have a path between them.

Theorem 3.1.3 *Let G be a simple graph with n-vertices and m components. Then G can have at most $\frac{1}{2}(n-m)(n-m+1)$ edges.*

Proof : Let the m components of G be $G_1, \ldots, G_m$ and let n_i = number of vertices in G_i for $i = 1, \ldots, m$. Then, $n_i \geq 1$ and $\sum_{i=1}^{m} n_i = n$.

Since G_i is a simple graph, the maximum number of edges in G_i is $\frac{1}{2}n_i(n_i-1)$. Hence the maximum number of edges in G is

$$= \frac{1}{2}\sum_{i=1}^{m} n_i(n_i-1) = \frac{1}{2}\sum_{i=1}^{m} n_i^2 - \frac{1}{2}\sum_{i=1}^{m} n_i = \frac{1}{2}\sum_{i=1}^{m} n_i^2 - \frac{1}{2}n \quad \ldots\ldots\ldots\ldots (1)$$

Now, for the positive integers $n_1, \ldots, n_m$ we have,

$$(n_1-1)+(n_2-1)+\cdots+(n_m-1) = \sum_{i=1}^{m} n_i - m = n - m \quad \ldots\ldots\ldots\ldots (2)$$

Squaring both sides of (2) we get, $(n_1-1)^2+\cdots+(n_m-1)^2+$

$2(n_1-1)(n_2-1)+2(n_1-1)(n_3-1)+\cdots = (n-m)^2$.

Hence, $(n_1-1)^2+\cdots+(n_m-1)^2 \leq (n-m)^2$, since $n_i - 1 \geq 0$, $i = 1, \ldots, m$.

That is, $\sum_{i=1}^{m} n_i^2 - 2\sum_{i=1}^{m} n_i + m \leq (n-m)^2$,

that is, $\sum_{i=1}^{m} n_i^2 \leq (n-m)^2 + 2n - m$. Hence from (1), the maximum

number of edges in G is $\leq \frac{1}{2}\{(n-m)^2+2n-m\} - \frac{1}{2}n$

$$= \frac{1}{2}\{(n-m)^2+2n-m-n\} = \frac{1}{2}\{(n-m)^2+(n-m)\}$$

$$= \frac{1}{2}(n-m)(n-m+1).$$

Theorem 3.1.4 *A connected graph with n vertices has at least n–1 edges.*

Proof : (by induction). The theorem is evidently true for n = 2. Suppose the theorem is true for any graph with number of vertices < n. Let now G be a connected graph with n vertices. Let v be an arbitrary vertex of G and let $G_1 = G-v$. Then the number of vertices of G_1 is n–1 and by the induction hypothesis its number of edges is ≥ n–2. Now v is adjacent to at least one vertex of G–v and so the number of edges of G = the number of edges of (G–v) + the number of edge(s) incident with v.

$\geq (n-2) + 1$

$= n-1.$

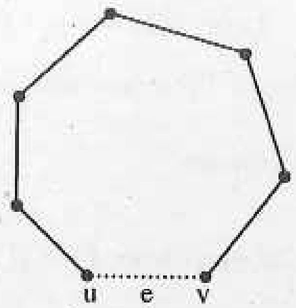
Hence the theorem is proved.

Definitions : Acyclic graph : A graph having no cycle is called an **acyclic** graph.

Cycle-edge of a graph : Let G be a graph. An edge e is called a **cycle-edge** if it is in some cycle of G.

Theorem 3.1.5 *A connected graph G remains connected after removing an edge e from G if and only if e is a cycle-edge in G.*

Proof : Let G be a connected graph and e be a cycle-edge in G. Since removal of a loop or a parallel edge from a graph does not change the connectivity of the graph, we may assume that e is not a loop or a parallel edge. We take a cycle C in G as shown below :



Let the cycle-edge e whose end points are u and v in C be deleted from G. We can reach v from u by travelling along the other edges of C in the graph G-e. Thus, there is a path between u and v and hence between any pair of vertices. Hence the graph G-e is also connected.

Conversely, let G-e be connected. Then there is a path P from u to v in G-e. Then, P+e forms a cycle in G containing e, that is, e is a cycle-edge.

Corollary 3.1.1 Let e be a cycle-edge of a graph G. Then, c(G-e) = c(G), where c(G) denotes the number of components of G.

3.2 Complete graph

Complete graph : (or, Universal graph) : A simple graph in which every pair of vertices is joined by an edge is called a **complete graph**.

A complete graph on n vertices is denoted by K_n. A complete graph on three vertices is a triangle.

Example 3.2.1 Complete graphs on one, two, three, four and five vertices are shown below in fig. 3.2.1.

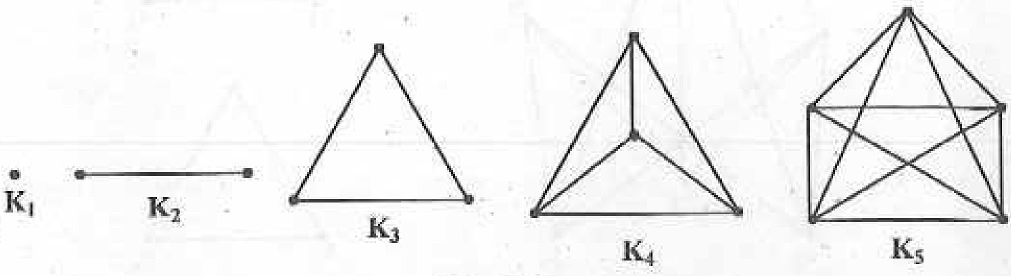


Fig. 3.2.1

Complement of a graph : Let G be a simple graph with n vertices. The **complement of G**, denoted $\overline{G}$, is the graph obtained from the complete graph K_n by removing the edges of G.

Observe that $\overline{G}$ is also simple and G and $\overline{G}$ have the same set of vertices. But two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Note : $\overline{K}_n$, the complement of the complete graph K_n, consists of n vertices and no edges, i.e., $\overline{K}_n$ is the null graph with n vertices. Again, complement of the null graph with n vertices is the complete graph K_n. Evidently, $\overline{\overline{G}}$ is isomorphic to G.

Example 3.2.2 A graph G and its complement $\overline{G}$ are shown below in the figures 3.2.2 and 3.2.3.

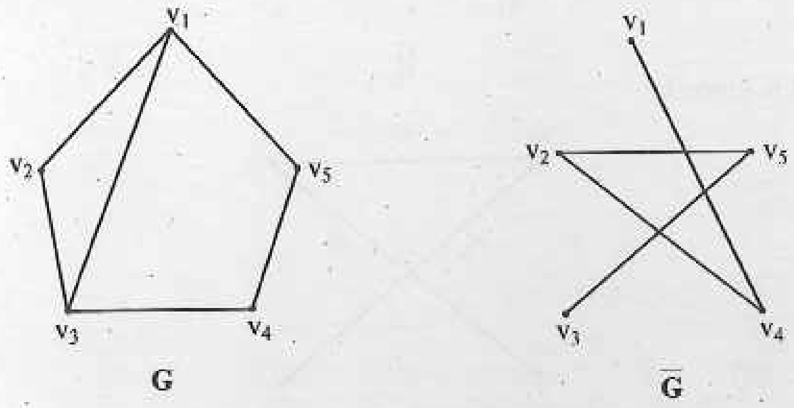
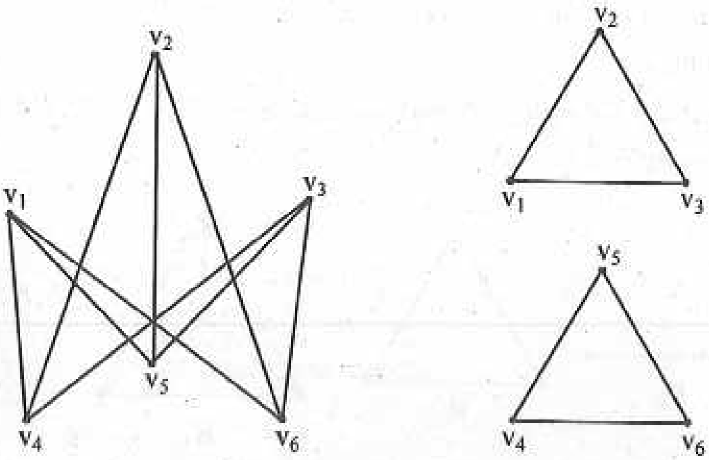


Fig. 3.2.2



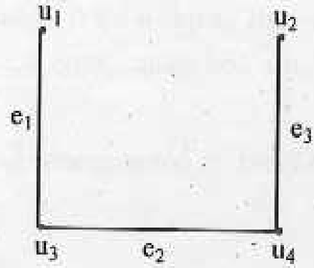
G $\qquad\qquad$ $\overline{G}$

Fig. 3.2.3

Observe that in fig. 3.2.2 both G and $\overline{G}$ are connected whereas in fig. 3.2.3 G is connected and $\overline{G}$ is disconnected.

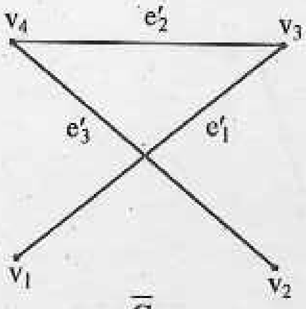
Self-complementary graph : A simple graph which is isomorphic to its complement is called **self-complementary.**

Example 3.2.3 Let G be the following graph :



G

Then, $\overline{G}$ is given by



$\overline{\mathbf{G}}$

We define $f : V_G \rightarrow V_{\overline{G}}$ by $f(u_i) = v_i$, $i = 1, \ldots\ldots, 4$ and

$\phi : E_G \rightarrow E_{\overline{G}}$ by $\phi(e_i) = e'_i$, $i = 1, 2, 3$.

We see that f and ϕ are one-to-one correspondences such that the incidence relation is preserved. Hence G and $\overline{G}$ are isomorphic and so G is self-complementary.

The problem of Ramsey : A well-known puzzle is as follows : Prove that at any party with six people it is always possible to find either three people who know each other, or, three people among whom no one knows any other. Acquaintance is assumed to be mutual.

Solution : The situation of the problem may be represented by a simple graph with six vertices. Each vertex represents a person and an edge in G joins two vertices if the two persons know each other. Then an edge in $\overline{G}$ implies that the persons representing its end points do not know each other. Thus the problem reduces to :

"For any simple graph G with six vertices, either G or $\overline{G}$ contains a triangle (i.e. a complete graph with three vertices)"

To prove it, let v be an arbitrary vertex of a simple graph G with six vertices. Then each of the other five vertices is adjacent to v either in G or in $\overline{G}$. We divide these five vertices into two groups such that one group contains those vertces which are adjacent to v in G and the other group contains those vertices which are adjacent to v in $\overline{G}$. One of the groups must contain at least three vertices. Without any loss of generality we may assume that there are three vertices v_1, v_2, v_3 adjacent to v in G. If any two of these vertices are adjacent, then they form a triangle whose third vertex is v. If no two of them are adjacent in G, then v_1, v_2, v_3 will form a triangle in $\overline{G}$.

Theorem 3.2.1 *A simple graph G and its complement $\overline{G}$ can not be both disconnected.*

Proof : If G is connected, there is nothing to prove. So, we assume G with n vertices to be disconnected. Let the components of G be $G_1, \ldots\ldots, G_m$, $m \geq 2$. We propose to show that $\overline{G}$ is connected. If $n = 2$, i.e. if G contains only two vertices

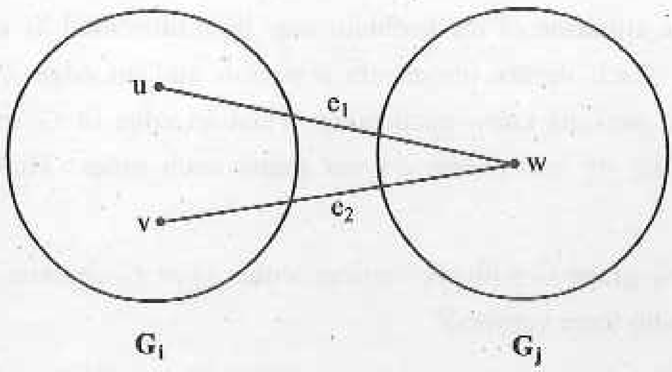
v_1, v_2, then there is no edge between v_1 and v_2 in G since G is disconnected. Hence, v_1 and v_2 must be joined by an edge in $\overline{G}$. So, $\overline{G}$ is connected.

We now assume that $n \geq 3$. Let u, v be two arbitrary vertices of G and hence of G. There are two possibilities.

Case 1 : Let $u \varepsilon G_i$, $v \varepsilon G_j$, $i \neq j$, $i, j = 1, \ldots, m$. Since u, v belong to different components of G, there is no edge between u and v in G.

Hence, u and v must be joined by an edge in $\overline{G}$.

Case 2 : Let $u, v \varepsilon G_i$, $i = 1, \ldots, m$. We take a vertex $w \varepsilon G_j$, $j \neq i$. Then there are no edges between u and w and between v and w in G.



Hence there must be an edge e_1 (say) between u and w and an edge e_2 (say) between v and w in $\overline{G}$. Thus, (u, e_1, w, e_2, v) forms a path between u and v in $\overline{G}$.

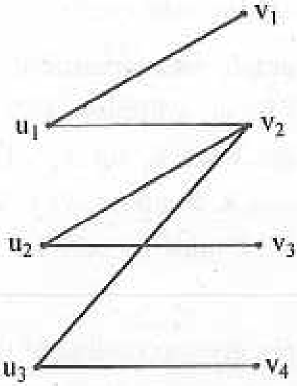
Thus, in any case, u and v are joined by a path in $\overline{G}$. Since u, v are two arbitrary vertices of $\overline{G}$, $\overline{G}$ is connected.

3.3 Bipartite graphs

A simple graph G is called a **bipartite graph** if its vertex-set V can be partitioned into two subsets V_1 and V_2 such that each edge in G has one end point in V_1 and the other end point in V_2.

The pair $\{V_1, V_2\}$ is called a bipartiton of G and V_1 and V_2 are called the bipartition subsets.

Example 3.3.1 The graph shown in fig. 3.3.1 is bipartite.



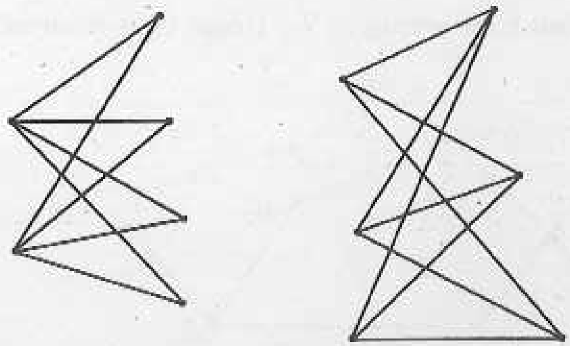
G

Fig. 3.3.1

Here the vertex-set $V = \{u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$ and the bipartition subsets are $V_1 = \{u_1, u_2, u_3\}$ and $V_2 = \{v_1, v_2, v_3, v_4\}$. Every edge in G has one end point in V_1 and the other end point in V_2.

Complete bipartite graph : A bipartite graph G with bipartition subsets V_1 and V_2 is called a **complete bipartite graph** if every vertex of V_1 is adjacent to every vertex of V_2. A complete bipartite graph G having m vertices in one bipartition subset and n vertices in the other bipartition subset is denoted by $K_{m,n}$. We then say that G is a complete bipartite graph on m and n vertices.

Example 3.3.2 The complete bipartite graphs $K_{2,4}$ and $K_{3,3}$ are shown below in fig. 3.3.2.



$K_{2,4}$ $K_{3,3}$

Fig. 3.3.2

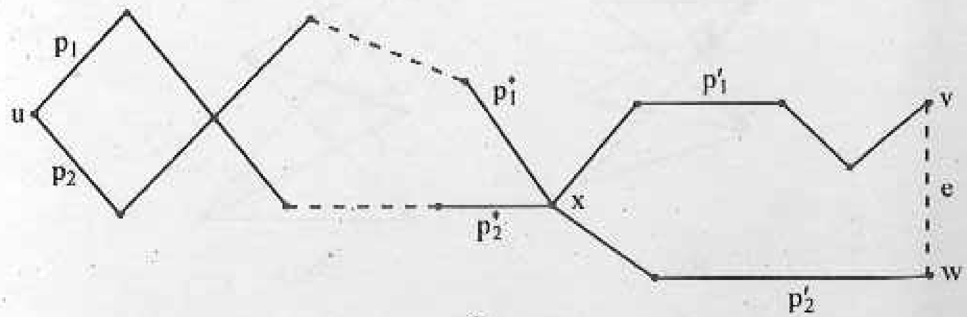
Theorem 3.3.1 *A simple graph is bipartite if and only if all its cycles are even. (In otherwords, it does not contain any odd cycle).*

Proof : Let G be a bipartite graph with bipartition subsets V_1 and V_2. Let C = $(v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1} = v_1)$ be an arbitrary cycle of length k in G. Since v_i and v_{i+1} are the end points of the edge e_i for i = 1, ..., k, it follows that if $v_i \varepsilon V_1$, then $v_{i+1} \varepsilon V_2$ for i = 1, ..., k. Suppose, $v_1 \varepsilon V_1$. Then $v_i \varepsilon V_1$ if and only if i is odd. Since $v_{k+1} = v_1 \varepsilon V_1$, k+1 must be odd; i.e., k is even. Hence the cycle C is even.

For the converse part, we simply give an outline of the proof. The detailed proof can be found in [5] or in [1] or in any standard book on graph theory.

Suppose, every cycle of G is even and components of G are $G_1, \ldots, G_m$ $(m \geq 1)$.

Show : each G_i is bipartite. For this, let u ε V_{Gi}. Define $V_1 \subset V_{Gi}$ by $V_1 = \{v \varepsilon V_{Gi} : d(u,v) \text{ is even}\}$ and $V_2 = V_{Gi} - V_1$. Then V_1 and V_2 form a partition of V_{Gi}. Let e be an edge with endpoints v, w and if possible, let v, w $\in V_1$. Take the shortest path p_1 in G_i from u to v and the shortest path p_2 in G_i from u to w. Then, p_1 and p_2 are of even lengths. Starting from u, let x be the last vertex common to p_1 and p_2. Let p_1*, p_2* be the subpaths of p_1 and p_2 respectively from u to x. Then, p_1*, p_2* are of equal lengths. Let p_1^1 be the subpath of p_1 from x to v and p_2^1 be the subpath of p_2 from x to w. Now, p_1 and p_2 are of even lengths implies the lengths of $p_1{}^1$ and p_2^1 are either both even or both odd. Then the walk W = (x, p_1^1, v, e, w, p_2^1, x) forms an odd cylce, a contradiction. Hence v, w cannot both belong to V_1. Similarly, they can not both belong to V_2. Hence G_i is bipartite which implies G is bipartite.



G_i

3.4 Worked Out Exercises

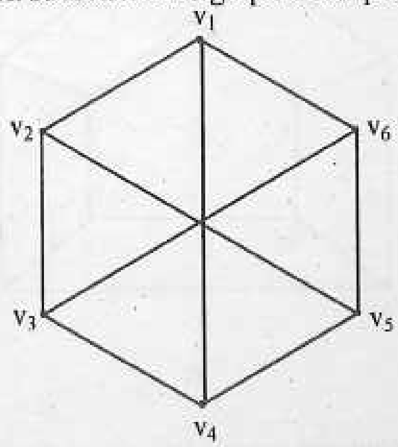
1. Let G be a connected graph with $n \geq 2$ vertices and m edges such that $m < n$. Prove that G has at least one pendant vertex.

Proof : Since G is connected, it has no isolated vertex. So, $d(v_i) \geq 1$ for every $v_i \in V_G$. If possible, let G have no vertex of degree 1. Then, $d(v_i) \geq 2$ for every $v_i \in V_G$. Since G has n vertices, $\sum_{i=1}^{n} d(v_i) \geq 2n$ and hence number of edges in G is $\geq \frac{1}{2}.2n = n$. That is, $m \geq n$. This contradicts the given condition $m < n$. Hence G contains at least one pendant vertex.

2. Show that the number of vertices of a self-complementary graph must be of the form 4p or, 4p + 1 where p is a non-negative integer.

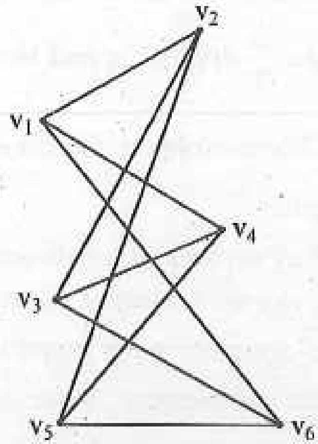
Solution : Let G be a self-complementary graph having n vertices and m edges. Then G is isomorphic to $\overline{G}$, its complement. Since the complete graph K_n contains $\frac{1}{2}n(n-1)$ edges, $\overline{G}$ contains $\frac{1}{2}n(n-1) - m$ edges. Since G and $\overline{G}$ are isomorphic, $\frac{1}{2}n(n-1) - m = m$, i.e., $n(n-1) = 4m$ ..... (1). If one of n and $n-1$ is even, then the other is odd. Hence (1) can hold only when either $n = 4p$ or, $n - 1 = 4p$; i.e., $n = 4p + 1$ where p is a non-negative integer.

3. Determine whether the following graph G is bipartite. If it is so, redraw the graph showing the bipartition subsets. Is the graph a complete bipartite graph?



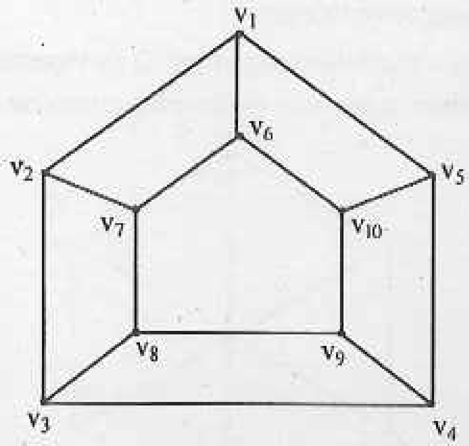
G

Solution : Let, if possible, G be bipartite with vertex-set $V = \{v_1, \ldots, v_6\}$ and bipartition subsets V_1 and V_2. Let $v_1 \in V_1$. Then, its adjacent vertices $v_2, v_4, v_6 \in V_2$. Since $v_2 \in V_2$, its adjacent vertices $v_3, v_5 \in V_1$. Thus if we take $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ we find that $V = V_1 \cup V_2$ and each edge of G joins a vertex in V_1 to a vertex in V_2. Hence G is bipartite.



G is a complete bipartite graph as evident from the redrawn graph.

4. Is the following 3-regular graph G bipartite? Justify your answer.



G

Solution : The graph contains a cycle $(v_1, v_2, v_3, v_4, v_5, v_1)$, which is odd. Hence it can not be bipartite.

3.5 Exercises

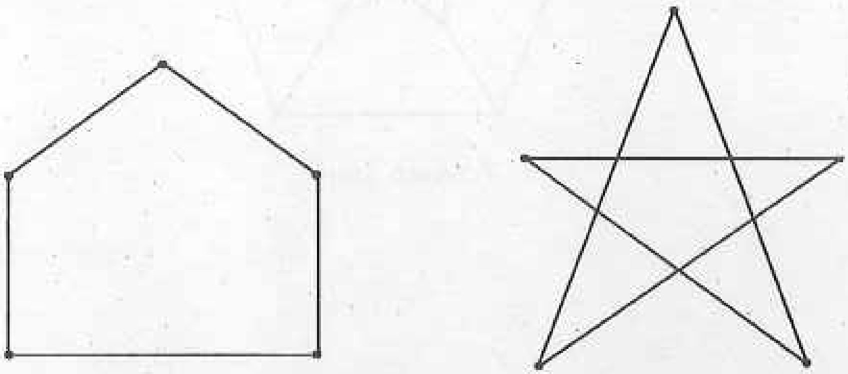
1. Let G be a complete graph with n verticles. Prove that G contains exactly $\frac{1}{2}n(n-1)$ edges.

2. Let G be a simple graph with at most 2n vertices. If $d(v_i) \geq n$ for every $v_i \varepsilon V_G$, prove that G is connected.

3. Prove that a simple graph with n ($\geq$2) vertices must be connected if it has more than $\frac{1}{2}(n-1)(n-2)$ edges.

4. Draw a connected graph that becomes disconnected when any edge is removed from it. Prove that such a graph must be simple and acyclic.

5. Prove that the following two graphs are self-complementary.

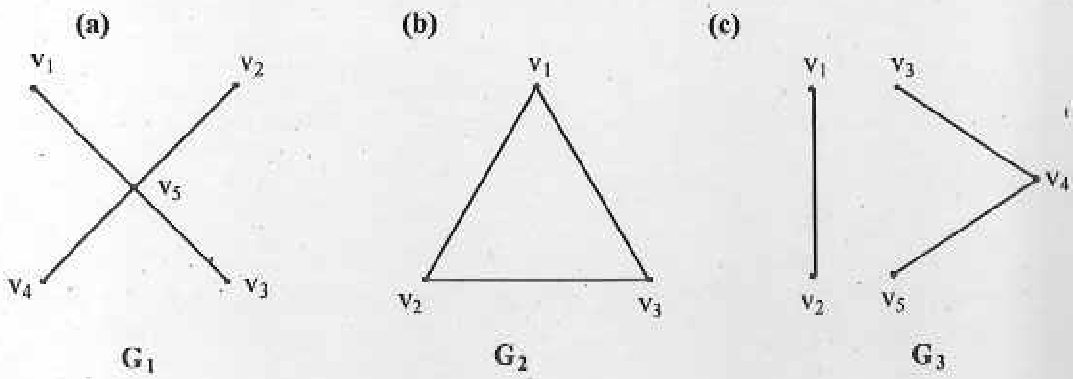


6. Draw the complement of the graph $K_{2,3}$.

7. How many edges are there in K_n and $K_{m,n}$?

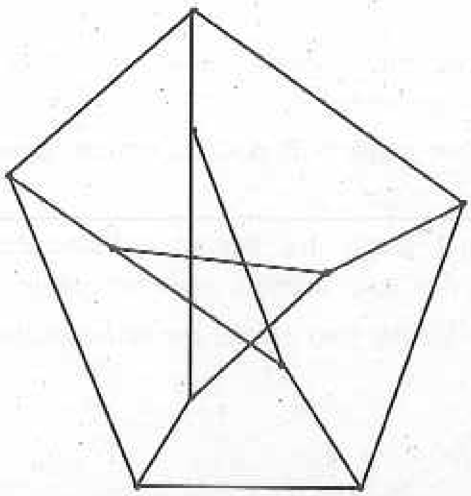
8. How many edges does the complement of a simple graph with n vertices and m edges have?

9. State which of the following graphs are bipartite. Justify your answer.



10. Draw a 3-regular bipartite graph.

11. The following graph is known as Petersen graph. Is it bipartite? Justify your answer.



Petersen graph

Unit 4 □ Eulerian and Hamiltonian Graphs

Structure

4.1 Euler Graphs

4.2 Hamiltonian Graphs

4.3 Worked Out Exercises

4.4 Exercises

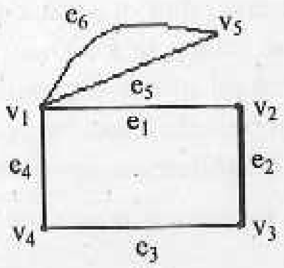
4.1 Euler Graph

Euler Trail : An open trail in a graph G is called an **Euler trail** if it contains all the edges of G.

Euler Circuit : A circuit in a graph G is called an **Euler circuit** if it contains all the edges of G.

If a graph G contains only one vertex v and no edge we call the walk (v) an Euler circuit.

Example 4.1.1. Consider the following graph G :

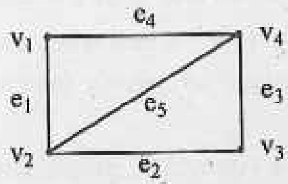


G

Fig 4.1.1

Then (v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1, e_5, v_5, e_6, v_1) is an Euler circuit in G.

Example 4.1.2. Consider the following graph G :



G

Fig 4.1.2

G has no Euler circuit, but it has an Euler trail. The open trail (v_2, e_5, v_4, e_3, v_3, e_2, v_2, e_1, v_1, e_4, v_4) is an Euler trail in G.

Edge-traceable graph : A graph which has an Euler circuit or an Euler trail is called an **edge-traceable graph**. Such a graph has the property that it can be drawn with a pencil without lifting the pencil from the paper and without tracing any edge twice.

Eulerian (or, Euler) graph : A graph G is said to be an **Eulerian graph** if it has an Euler circuit.

The graph given in 4.1.1 is Eulerian. A graph consisting of a single vertex and no edge is treated as Eulerian.

Theorem 4.1.1 *If a connected graph G is Eulerian, then every vertex of G is of even degree.*

Proof : Let G be a connected Eulerian graph. If G consists of only one vertex v and no edge, then d(v) = 0 (even). If G contains only one vertex v and a finite number of loops at v, then d(v) = even, since every loop contributes 2 to the degree of v.

Now, we assume that G contains more than one vertex. Since G is Eulerian, it has an Euler circuit, say C = (v_1, e_1, v_2,....., v_k, e_k, v_{k+1} = v_1). Let u be an arbitrary vertex of G. Then, u must be an end point of some edge since G is connected. Since C contains all the edges, u must belong to C. Then for each appearance of u in C there are two new edges incident on u, one for entering u and another for exiting from u since no edges in C are repeated. These two edges contribute 2 to the degree of u. Thus, d(u) is even. Since u is arbitrary, every vertex of G is of even degree.

Theorem 4.1.2 : *If G is a connected graph and every vertex of G is of even degree, then G is Eulerian.*

Proof : Let G be a connected graph and every vertex of G be of even degree. We prove the theorem by induction on the number of edges n of G.

Basis step : Let n = 0. Since G is connected and has no edge, G consists of a single vertex. Hence, by definition, G is Eulerian. Next let n = 1. By the condition of the theorem, G is a loop at a vertex and so is Eulerian

Inductive Hypothesis : Let n be a positive integer and we assume that any connected graph with k (<n) edges in which every vertex is of even degree, is Eulerian.

Induction step : Let G be a connected graph with n edges and each vertex of G be even. Since each vertex is of degree at least two, G contains a circuit (cf : theorem 2.3.1), C say. If C contains all the edges of G, it becomes an Euler circuit and so G is Eulerian. If not, we remove from G all the edges of C and obtain a subgraph G' of G. G' may not be connected, but each of its components will be connected and will contain fewer than n edges. Now, the removal of the edges of C either leaves the degree of a vertex unchanged or reduces the degree by two. Hence, all the vertices of each component of G' will be of even degree. Therefore, by the inductive hypothesis each component of G' is Eulerian. Moreover, since G is connected, each component of G' must meet the circuit C at least at one vertex. To show that G has an Euler circuit, we construct such a circuit in G as follows : We start at a vertex u (say) of C and traverse C until we meet a vertex v_1 (say) of one of the components of G'. We then traverse Euler circuit of that component and return to the circuit C at the same vertex v_1. We then continue along C and traverse each component of G' as it meets the circuit C. Eventually, we can traverse all the edges of G exactly once and come back at u. This produces an Euler circuit in G. Hence G is Eulerian. Thus, by the induction principle, any connected graph all of whose vertices are of even degree is an Eulerian graph.

Note 1. The process of producing an Euler circuit as given in the proof is illustrated in the following graph :

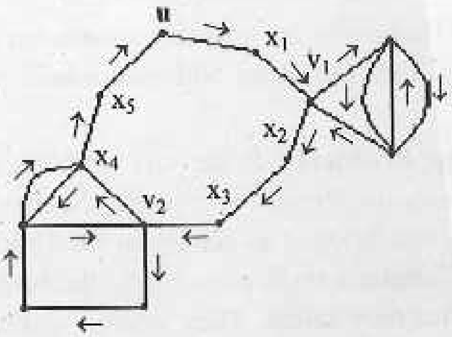


Fig 4.1.3

In the above graph C is the cycle (u, x_1, v_1, x_2, x_3, v_2, x_4, x_5, u).

Note 2. Combining theorems 4.1.1 and 4.1.2 we have the Euler's theorem as follows : "A connected graph G is Eulerian if and only if all the vertices of G are of even degree."

Fleury's algorithm : To find an Euler circuit in an Eulerian graph G the following algorithm, known as fleury's algorithm, is useful.

Fleury's algorithm : Step 1. Choose a vertex u of V_G as the starting vertex for the Euler circuit C.

Step 2. Traverse any available edge. But do not choose an edge if deleting it disconnects the remaining graph unless there is no alternative.

Step 3. After traversing each edge, remove it from E_G together with any isolated vertex that results.

Step 4. If no edges remain, stop. The Euler circuit has been obtained. Otherwise, select another available edge and repeat step 2 until no edges remain in E_G

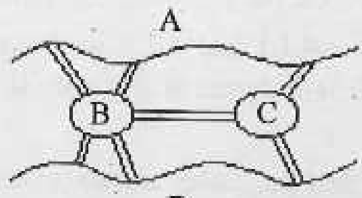
Theorem 4.1.3 *A connected graph G has an Euler trail if and only if it has exactly two odd-degree vertices.*

Proof : Let G be a connected graph having an Euler trail. Let u and v be the initial and final vertices of the Euler trail. If we add an edge e joining u and v in G we get a new connected graph G' which will have an Euler circuit. Then every vertex of G' will be of even degree. In particular, u and v in G' becomes even. But G and G' have the same set of vertices. If we now obtain G by removing the edge e we find that u and v are the only vertices of odd degree.

Conversely, let G be connected and have exactly two odd vertices u and v. By adding an edge e joining u and v in G we obtain a connected graph G' all of whose vertices are even. Hence there exists an Euler circuit in G'. If we now remove the edge e from this circuit we get an Euler trail in G.

Remark 4.1.1 To find an Euler trail in a connected graph with exactly two odd vertices we must begin the trail at an odd vertex and terminate it at the other odd vertex.

Königsberg Bridge Problem : In the city of Königsberg (renamed Kaliningrad in Russia) two islands in the Pregel river were connected to each other and to the outer river banks by seven bridges as shown in the following figure 4.1.4. The city people tried their best to take a walk crossing all the bridges exactly once and return to the starting point. But they failed. They wanted to know whether such a walk is possible.

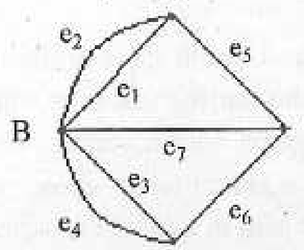


Königsberg Bridge Problem

Fig 4.1.4

Here A, D are the outer river banks and B, C are the islands. The problem is to start at any of the land areas A, B, C or D, walk over each of the seven bridges exactly once and return to the starting point. This seemed to be impossible, but no one could explain why. Swiss mathematician Leonhard Euler (1707-'83) proved that such a walk is impossible, and presented the solution in a paper in 1736.

Euler's model for the problem is a graph with four vertices and seven edges where the vertices represent the land areas A, B, C, D and the edges $e_1, e_2, \ldots, e_7$ represent the seven bridges. Then the configuration is represented by the following graph G :

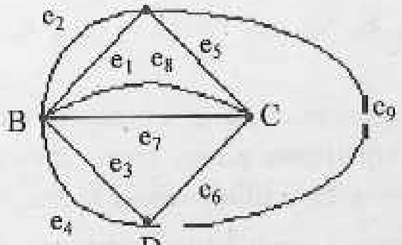


G

Fig 4.1.4

Now G is connected and crossing a bridge corresponds to traversing an edge of G. Then the problem is reduced to determining whether G is Eulerian. Since G has vertices of odd degree there is no Euler circuit in G, i.e., G is not Eulerian. Hence it is impossible to find a route to walk over all the seven bridges exactly once and return to the starting point.

Remark 4.1.2 At present there are two more bridges on the Pregel river, one connecting the land areas A and D, and the other connecting the islands B and C. The present configuration of the city is represented by the following graph G_1 :



G_1

Fig 4.1.6

The graph G_1 is connected and every vertex of G_1 is even. Hence G_1 has an Euler circuit. So, it is now possible to find a route to walk over all the nine bridges exactly once and return to the starting point. An Euler circuit in G_1 is :

(A, e_2, B, e_8, C, e_7, B, e_4, D, e_6, C, e_5, A, e_9, D, e_3 B, e_1, A).

4.2 Hamiltonian Graphs

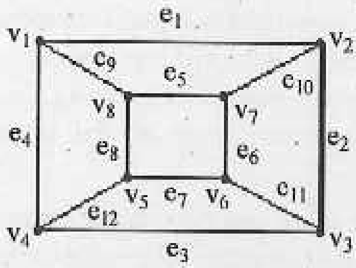
Hamiltonian Path : A path in a graph G that contains every vertex of G is called a **Hamiltonian Path.**

Hamiltonian Cycle : A cycle in a graph G that contains every vertex of G is called a **Hamiltonian cycle.**

Thus, a Hamiltonian cycle in a graph G is a closed walk that traverses every vertex of G exactly once, except the starting vertex at which the walk also terminates. Hence a Hamiltonian cycle in a graph with n vertices consists of exactly n edges. If we remove any one edge from a Hamiltonian cycle, we get a Hamiltonian path. Hence the length of a Hamiltonian path in a simple n-vertex graph is n-1.

Hamiltonion Graph : A graph having a Hamiltonian cycle is called a **Hamiltonian graph.** Observe that a Hamiltonian graph must be connected.

Example 4.2.1 The following graph G as shown in fig. 4.2.1 is Hamiltonian.



G

Fig 4.2.1

The cycle C = (v_1, e_9, v_8, e_8, v_5, e_{12}, v_4, e_3, v_3, e_{11}, v_6, e_6, v_7, e_{10}, v_2, e_1, v_1) is a Hamiltonian cycle.

Remark 4.2.1 Any Hamiltonian cycle in a bipartite graph must have the same number of verticles in each bipartition subset, since any edge in a bipartite graph corresponds to a move from one bipartition subset to the other.

No elegant characterisation is known for the existence of a Hamiltonian cycle in a connected graph. This is an unsolved problem in graph theory, although several necessary or sufficient conditions are known.

Theorem 4.2.1 *Let G be a Hamiltonian graph with vertex set V. For every non-empty proper subset S of V, c (G-S) ≤ n(S), where c(G) denotes the number of components of G and n(S) denotes the number of vertices in S.*

Proof : Let G be a Hamiltonian graph with vertex-set V, K be a Hamiltonian cycle in G and $\phi \neq S \subset V$. Now, if we delete the set S of vertices together with the edges incident with them from G then they will also be deleted from the cycle K, since K contains all the vertices of G. Consequently, the cycle K will be divided into at most n(S) pieces, i.e., c(K-S) ≤ n(S). But K-S is a subgraph that contains every vertex of G-S. Hence, G-S cannot have more components than K-S, i.e, c(G-S) ≤ c(K-S). Therefore, c(G-S) ≤ n(S).

Note : The condition is not sufficient. However, we can use this condition to show that a certain graph is not Hamiltonian.

Theorem 4.2.2 *(Dirac, 1952) : Let G be a simple connected graph with $n \geq 3$ vertices. If $d(v) \geq \frac{n}{2}$ for every vertex $v \in V_G$, then G is Hamiltonian.*

Proof : We omit the proof. However, the proof can be found in [1,p 655].

We now state another sufficient condition for a graph to be Hamiltonian.

Theorem 4.2.3 *[Ore, 1960]. Let G be a simple connected n-vertex graph where $n \geq 3$. If for every pair of non-adjacent vertices $u, v \in V_G$, $d(u) + d(v) \geq n$, then G is Hamiltonian.*

Proof : We omit the proof. The proof can be found in [3, p 223]

4.3 Worked Out Exercises

1. Prove that a complete graph K_n is Eulerian if and only if n is odd.

Proof : The complete graph K_n is connected and contains n vertices. Each vertex of K_n is joined to the remaining n-1 vertices by edges. Since K_n is simple, there are no parallel edges and loops. Hence, $d(v_i) = n-1$ for each vertex v_i, $i = 1, \ldots\ldots, n$. Hence, each $d(v_i)$ will be even if and only if n-1 is even, i.e., if and only if n is odd. Hence, K_n will be Eulerian if and only if n is odd.

2. Let G be a simple connected graph with $n \geq 3$ vertices and m edges. If $m \geq \frac{1}{2}(n-1)(n-2)+2$, show that G is Hamiltonian.

Solution : Let G be a simple connected graph with $n \geq 3$ vertices and m edges such that $m \geq \frac{1}{2}(n-1)(n-2)+2$. If possible, let G be not Hamiltonian. Then, by

Theorem 4.2.3, there must exist two non-adjacent vertices u, v in G such that $d(u) + d(v) \le n - 1$. There are, therefore, at most $n - 1$ edges in G which are incident with either u or v. But in the complete graph K_n, number of edges incident with any two vertices is $(n - 1) + (n - 2) = 2n - 3$. Hence, the difference between the number of edges incident with u and v in K_n and the number of edges incident with u and v in G is at least

$2n - 3 - (n - 1) = n - 2$. Hence the number of edges in G is at most

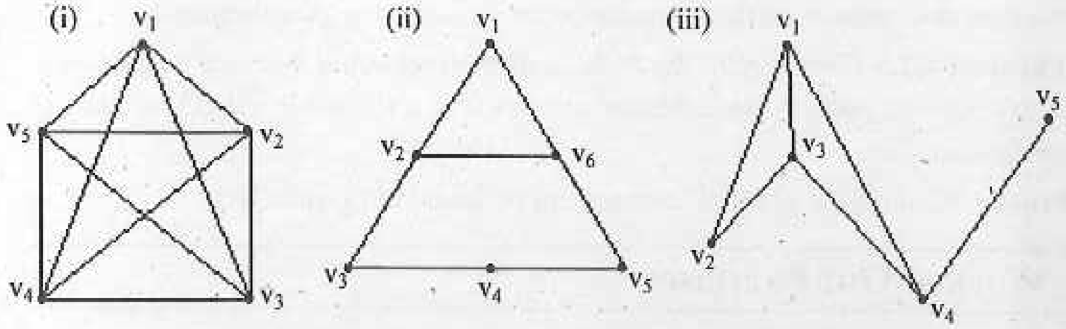
$$\frac{n(n-1)}{2} - (n-2) = \frac{1}{2}n(n-1) - (n-1) + 1 = \frac{1}{2}(n-1)(n-2) + 1.$$

That is, $m \le \frac{1}{2}(n-1)(n-2) + 1$ which contradicts the given condition.

Hence, G is Hamiltonian.

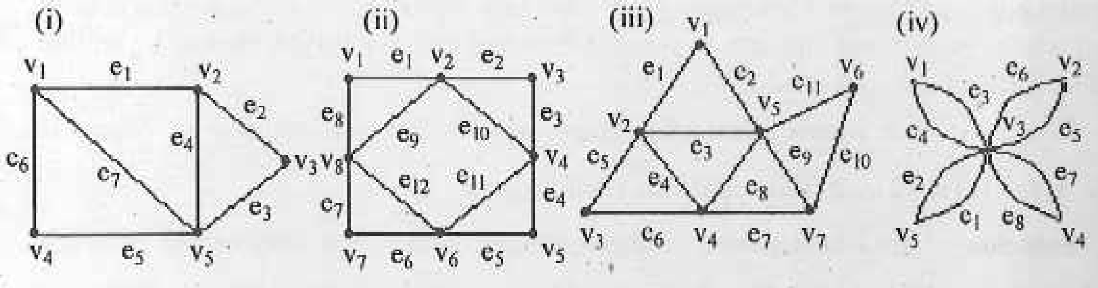
4.4 Exercises

1. For the following three graphs can you trace all the edges with a pencil without taking the pencil off the paper and without going through any edge twice? Justify your answer.



G_1 G_2 G_3

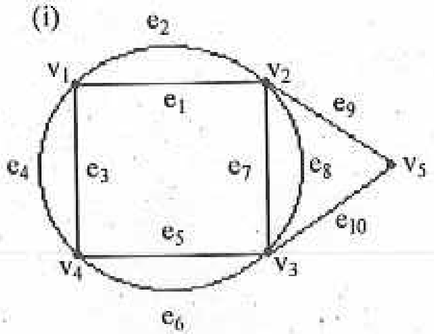
2. Determine whether each of the following graphs has an Euler circuit. Find an Euler circuit, if it exists.



G_1 G_2 G_3 G_4

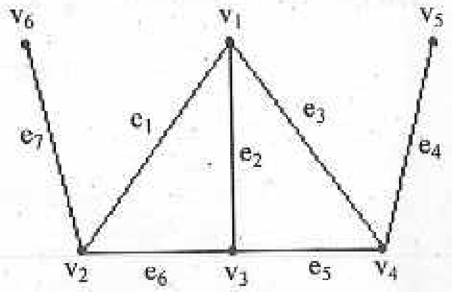
3. Determine whether each of the following graphs has an Euler trail. Find an Euler trail, if it exists.

(i)



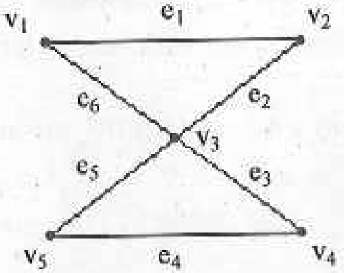
G_1

(ii)



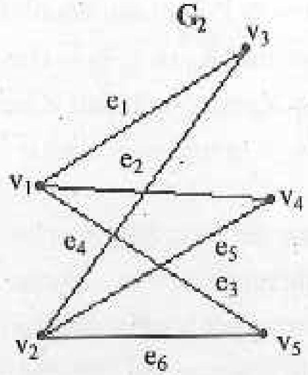
G_2

(iii)



G_3

(iv)

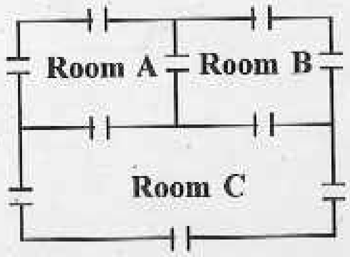


G_4

4. When is a complete bipartite graph $K_{m,n}$ Eulerian? Justify your answer.

5. Prove that a connected graph G is Eulerian if and only if the set of edges can be partitioned into cycles.

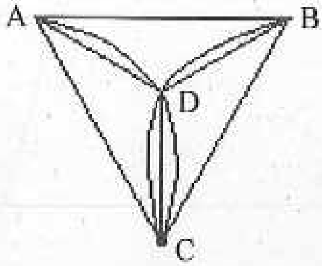
6. Consider the floor plan of the three-room flat with doors as shown below.



D (outside)

Each room is connected by doors with every other room and outside as shown in the plan. Is it possible to start in a room or outside and take a walk that goes through each door exactly once? If there is such a walk, find it. Can you return to the starting point through this walk? Justify your answer.

Hints : The corresponding graph is :



There exists Euler trail, but no Euler circuit.

7. Prove that K_n $(n \geq 3)$ is Hamiltonian.

8. Draw a graph such that it has a Hamiltonian path but no Hamiltonian cycle.

9. Draw a graph such that it has neither a Hamiltonian path nor a Hamiltonian cycle.

10. Draw the following graphs with the given properties and justify your answer.

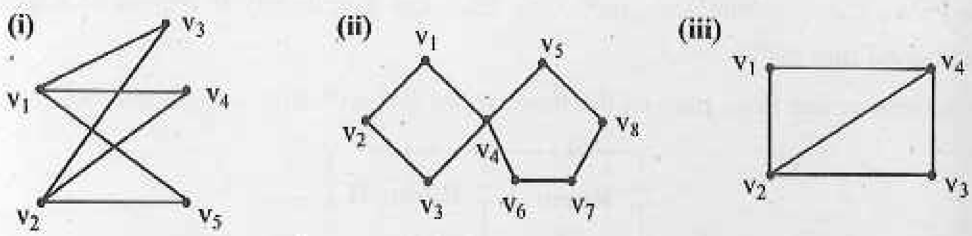
(i) Hamiltonian but not Eulerian.

(ii) Eulerian but not Hamiltonian.

(iii) Both Eulerian and Hamiltonian.

(iv) Neither Eulerian nor Hamiltonian.

11. Are the following graphs Hamiltonian? Justify your answer.



Unit 5 □ Tree

Structure

5.1 Tree

5.2 Spanning Tree

5.3 Rooted Tree

5.4 Binary Tree

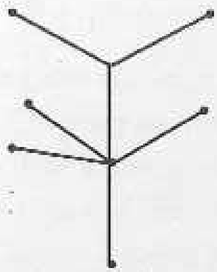
5.5 Worked Out Exercises

5.6 Exercises

5.1 Tree

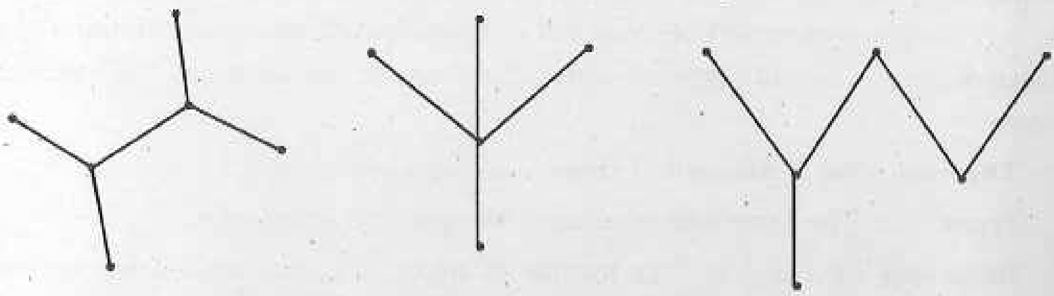
A connected graph having no cycle is called a **tree**. Since a loop or parallel edges form a cycle, a tree must be a simple graph.

For example, the following graph is a tree.



Tree

Forest : A graph without any cycle is called a **forest**. Thus the components of a forest are trees. For example, the following graph is a forest.



Forest

We now consider a few properties of a tree in the form of the following theorems.

Theorem 5.1.1 *Every pair of vertices in a tree is joined by a unique path.*

Proof : Let T be a tree and v_i, v_j be any two vertices of T. Since T is connected, there is at least one path between v_i and v_j. If possible, let there be two distinct paths p_1, and p_2 between v_i and v_j. The union of these two paths will contain a cycle in T, which is a contradiction since T can not contain any cycle. Hence the path joining v_i and v_j is unique.

Theorem 5.1.2 *Let G be a graph such that it has a unique path between every pair of vertices. Then G is a tree.*

Proof : Evidently, G is connected. If possible, let G contain a cycle. Then there exists at least one pair of vertices v_i and v_j such that there are two distinct paths between them. This is a contradiction to the given condition. So G is a tree.

Note : Combining theorems 5.1.1 and 5.1.2 we can say, "A graph G is a tree if and only if there is a unique path between every pair of vertices in G."

Theorem 5.1.3 *A tree with more than one vertex contains at least two pendant vertices i.e., vertices of degree 1.*

Proof : Let T be a tree and v_i, v_j be any two vertices of T. Then there is a unique path between v_i, v_j. Let L be the set of all paths in T. Then L is finite since the vertex-set of T is finite. Hence we can find a path P in T with maximal number of vertices. Let the path P be from the vertex u to the vertex v. Then, u and v will be pendant vertices. First we show that u is a pendant vertex. If possible, let $d(u) > 1$. Then u has more than one adjacent vertices. One of them, say u_1, must lie on P. Then another adjacent vertex of u, say u_2, can not lie on P. For, in that case, from u we may go to u_1 and then to u_2 along P and come back to u again along the other edge, say e, joining u and u_2, thus forming a cycle. Thus, u_2 is not a member of P. Then the path (u_2, e, P) contains more vertices than that of P which contradicts the definition of P. Hence $d(u) = 1$, i.e., u is a pendant vertex. Similarly we can show that v is a pendant vertex.

Theorem 5.1.4 *A tree with n vertices contains exactly $n - 1$ edges.*

Proof : Let T be a tree with n vertices. We apply induction on n.

Basis step : Let $n = 1$. Then number of edges in T is o, since T has no loop which is a cycle. Thus the theorem is true when $n = 1$.

Induction Hypothesis : We assume that the result is true for any tree with m vertices, $m \geq 1$.

Induction Step : Let T be a tree with m + 1 vertices. Then T has a pendant vertex, say u. We delete u and the edge incident on it from T and let the new graph be T_1. Since T is connected and acyclic, T_1 is also connected and acyclic. Hence T_1 is a tree with m vertices and hence by induction hypothesis T_1 contains exactly m – 1 edges and so T contains (m – 1) + 1 = m edges. The result now follows by induction.

Theorem 5.1.5 *A connected n-vertex graph with n – 1 edges is a tree.*

Proof : Let G be a connected graph with n vertices and n – 1 edges. Suppose, if possible, G contains a cycle C. If we remove a cycle-edge from C, the resulting graph will be a connected graph with n vertices and n – 2 edges. This is a contradiction to the fact that a connected n-vertex graph contains at least n – 1 edges (cf. theorem 3.1.4). Hence G does not contain any cycle, i.e., G is a tree.

Note : Combining theorems 5.1.4 and 5.1.5 we can say, "A connected n-vertex graph is a tree if and only if it contains exactly n – 1 edges."

Theorem 5.1.6 *A forest with n vertices and m components contains exactly n-m edges.*

Proof : Let G be a forest with n vertices and m components G_1.............G_m. Let the component G_i contain n_i vertices, i = 1, ..., m. Then, $\sum_{i=1}^{m} n_i = n$. Each G_i is connected and acyclic. So, each G_i is a tree.

By theorem 5.1.4, $e(G_i) = n_i - 1$,

where $e(G_i)$ denotes the number of edges in G_i, i = 1, ...,m.

Now, $e(G) = \sum_{i=1}^{m} e(G_i) = \sum_{i=1}^{m} (n_i - 1) = \sum_{i=1}^{m} n_i - m = n - m$.

Theorem 5.1.7 *A graph T with n vertices is a tree if and only if T is acyclic and contains exactly n –1 edges.*

Proof : First we assume that T is a tree with n vertices. Then, by definition, T is acyclic. By theorem 5.1.4, T contains exactly n – 1 edges.

Conversely, let the graph T with n vertices be acyclic and contain exactly n – 1 edges. Suppose the graph T has m components. Then by theorem 5.1.6, the forest T contains n – m edges. So, m = 1, i.e., T is connected. Thus, T is a connected acyclic graph i.e., T is a tree.

Minimally Connected Graph : A connected graph G is said to be **minimally connected** if G becomes disconnected when we delete any one edge from G.

Theorem 5.1.8 *A graph is a tree if and only if it is minimally connected.*

Proof : Let T be a tree. Suppose, if possible, T is not minimally connected. Then there must exist at least one edge e (say) in T such that T – e is also connected. Then, by theorem 3.1.5 e must be a cycle-edge in T; i.e., T has a cycle which contradicts the fact that T is a tree. Hence T is minimally connected.

Conversely, let T be a minimally connected graph. We claim that T is acyclic. For, if T contains a cycle, we can delete an edge from the cycle and still T remains connected so that T will not be minimally connected. Thus, T is a tree.

Theorem 5.1.9 *A connected graph G is a tree if and only if adding an edge e between any two vertices in G creates exactly one cycle in G + e.*

Proof : Let G be a tree and u, v be any two vertices in G. Then there is exactly one path between u and v in G. If we add an edge e between u and v, an additional path is created between them and consequently a cycle is created in G + e. Since there was only one path between u and v before the addition of the edge e, there will be exactly one cycle in G + e after the addition of e.

Conversely, we assume that adding an edge e between any two vertices u and v in G creates exactly one cycle in G + e. Then, before the addition of e, there must have one and only one path between u and v in G. Hence G is a tree.

Note : From the previous theorems we can characterize a tree as follows :

Theorem 5.1.10 *Let T be a graph with n vertices. Then the following statements are equivalent.*

1. T is a tree.

2. Any two vertices of T are connected by exactly one path.

3. T is connected and has exactly n – 1 edges.

4. T is acyclic and has exactly n – 1 edges.

5. T is a minimally connected graph.

6. T is connected and adding an edge e between any two vertices in T creates exactly one cycle in T + e.

5.2 Spanning Tree

Let G be a graph and T be a subgraph of G. T is called a **spanning tree** of G if T is a tree which contains all the vertices of G.

Example 5.2.1 Consider the following graphs :

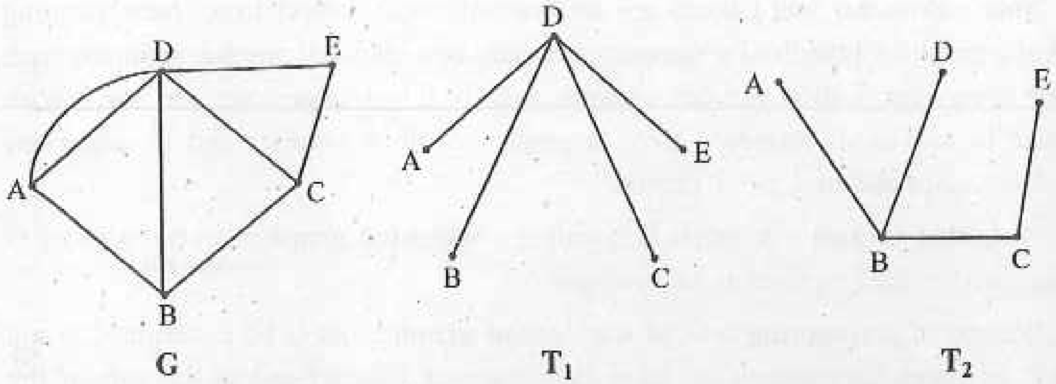


Fig. 5.2.1

In fig. 5.2.1 the subgraphs T_1 and T_2 of G are two different spanning trees of the graph G.

Note : We define a spanning tree only for a connected graph G. Because, if G is not connected, then there can not be any connected subgraph of G containing all the vertices of G.

Theorem 5.2.1 *Every connected graph has at least one spanning tree.*

Proof : Let G be a connected graph. If G is acyclic, then it is its own spanning tree. So, we assume that G contains at least one cycle. Let C be a cycle in G. We delete an edge from C and the new graph G' will remain connected and will contain all the vertices of G. If G contains more cycles, we repeat the deletion operation until an edge from the last cycle in G is removed. The resulting subgraph will be connected, acyclic and will contain all the vertices of G. Thus the resulting graph is a spanning tree of G.

Theorem 5.2.2 *A graph having a spanning tree must be connected.*

Proof : Let the graph G have a spanning tree T and u, v be any two vertices of G. Then u, v are also vertices of T. Since T is a tree, there is a unique path P between u and v in T. But T is a subgraph of G. Hence the path P from u to v must also be in G. Hence G is connected.

Note : Combining theorems 5.2.1 and 5.2.2 we can say, "A graph G has a spanning tree if and only if G is connected."

Branch and Chord : Let T be a spanning tree of a graph G. An edge of T is called a *Branch* of T. An edge of G that is not included in T is called a **chord** of T.

Note : Branches and Chords are defined only with respect to a given spanning tree of a graph. A branch of a spanning tree may be a chord of another spanning tree. Since every tree with n vertices contains exactly n – 1 edges, we can say, "With respect to any of its spanning trees, a graph G with n vertices and m edges has n – 1 branches and m – n + 1 chords."

Weighted Graph : A graph G is called a **weighted graph** if every edge of G is assigned a number, called its edge-weight.

Weight of a spanning tree in a weighted graph : Let G be a weighted graph and T be a spanning tree of G. Then the weight of T is defined as the sum of the weights of all the branches of T. Different spanning trees of a weighted graph may have different weights. Generally, among all the spanning trees of a weighted graph, the spanning tree with the minimum weight is of practical importance. However, there may have several spanning trees of a weighted graph with minimum weight.

Minimal Spanning Tree : Let G be a connected weighted graph. A spanning tree of G having the smallest weight is called a **minimal spanning tree** of G.

Application—We give below a problem to show the application of minimal spanning tree of a weighted graph.

Problem : Suppose that several towns are to be linked through a network of roads. The cost of building a direct road between a pair of towns is known for each possible pair. Now the problem is to find the least expensive network of roads that connects all the towns.

Solution : Let there be n towns. We label them as vertices $v_1, \ldots\ldots\ldots, v_n$ of a graph G and the direct roads between any two possible towns as the edges of G. There may be pair of towns between which a direct road cannot be constructed. In otherwords, there may be pair of vertices between which there is no edge. The cost of building a direct road between two towns is taken as the weight of the edge joining them. Clearly, the network of roads is a weighted connected graph G. The required minimum-cost i.e., minimum-weight network must be a spanning tree of G. Otherwise, we can remove an edge from a cycle and yet get a connected graph whereby the total cost is reduced. The desired network of roads must be a minimal spanning tree of G. Thus the problem reduces to : To find a minimal spanning tree of a connected weighted n-vertex graph.

There are several algorithms for finding a minimal spanning tree of a connected weighted graph. We describe below Kruskal's algorithm for such a tree, named after J. B. Kruskal.

Kruskal's algorithm for a minimal spanning tree : Let G be a connected weighted n-vertex graph. Kruskal's algorithm for finding a minimal spanning tree of G is as follows :

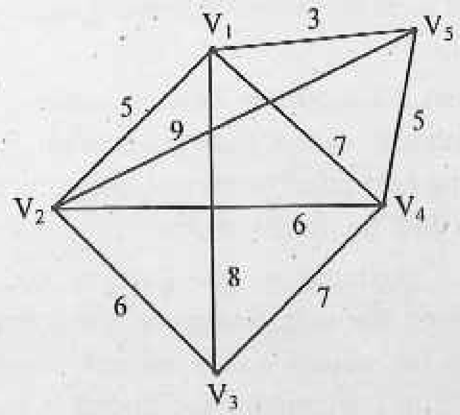
Step 1. List the weights of all the edges of G in non-decreasing order.

Step 2. First choose a minimum-weight edge of G. If there are several edges with minimum weight, choose any one of them.

Step 3. Select another edge whose weight is minimum among the remaining edges, provided the selected edge does not form a cycle with the already selected edges.

Step 4. Continue the process of selecting until n – 1 edges have been selected, because the required tree will contain n – 1 edges. Then stop. All the selected edges will form the desired minimal spanning tree. We illustrate the algoritham by the following example.

Example 5.2.1 For the following connected weighted graph G, apply Kruskal's algorithm to find a minimal spanning tree of G.



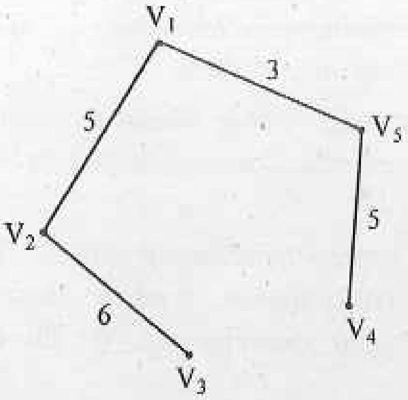
G

We proceed as follows :

Step 1. The weights of all the edges of G in non-decreasing order are 3, 5, 5, 6, 6, 7, 7, 8, 9.

Step 2. The smallest weight is 3 between v_1 and v_5. So we choose this edge $v_1 v_5$. (We denote the edge joining the vertices v_i and v_j by $v_i v_j$).

Step 3. The next smallest weight among the remaining edges is 5 between either v_4 and v_5 or, between v_1 and v_2. We choose both the edges v_1v_2 and v_4v_5 as the chosen edges will not form a cycle. The next smallest weight among the remaining edges is 6 between either v_2 and v_3 or, between v_2 and v_4. We cannot choose the edge between v_2 and v_4 as this would create the cycle $v_1\ v_2\ v_4\ v_5\ v_1$. But we can choose the edge between v_2 and v_3 as this will not create a cycle with the already selected edges.



Step 4. The number of vertices in G is five. So, the desired minimal spanning tree will have four edges. We have already selected four edges. So we stop. These four selected edges will form the desired minimal spanning tree of G, the total weight of the tree being 19 units.

Remark 5.2.1 Kruskal's algorithm is an example of what is known as *Greedy algorithm*. This algorithm is called greedy algorithm because at each step in the process of selection, the best possible current choice is taken and we do not think how that choice will affect the future choices.

We can also find a maximum-weight spanning tree for a connected weighted graph in a similar manner. The only difference is that first we choose the edge with largest weight and then the second-largest one and so on. There is another popular algorithm, known as Prim's algorithm, for finding a minimal spanning tree of a connected weighted graph. This can be found in [1] or [3].

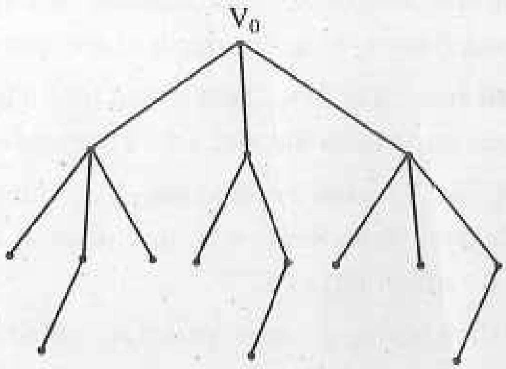
5.3 Rooted Tree

Directed Tree : A digraph whose underlying graph is a tree is called a **directed tree**.

Rooted Tree : A directed tree in which one particular vertex is distinguished from all other vertices, designated as the **root**, such that there is a unique directed path from the root to any other vertex is called a **rooted tree**.

We sometimes write $T(v_0)$ to indicate a rooted, tree T with the root v_0. In pictorial representation of a rooted tree we generally draw the root at the top of the graph, unlike our natural trees. Moreover, we omit the arrows from the arcs on the tacit understanding that the direction on the arc is always away from the root.

Example 5.3.1. The following graph is a rooted tree with v_0 as the root.



$T(V_0)$

Fig. 5.3.1

Theorem 5.3.1 *A directed tree T can be represented as a rooted tree with root v_0 if and only if v_0 has in-degree 0 and all other vertices of T have in degree 1.*

Proof : Let T be a rooted tree with root v_0. If possible, let there be an arc e coming from some vertex u to v_0. Then the directed path from v_0 to u together with e will form a cycle in the underlying graph of T. This contradicts the definition of T and hence v_0 has in-degree 0. Since there is a directed path from v_0 to all other vertices, every other vertex must have in-degree at least one. If possible, let some vertex w have in-degree more than one. Then there are at least two distinct vertices v_1 and v_2 as tails of two arcs e_1 and e_2 respectively with the same head w. Let P_1 and P_2 be the unique directed paths from the root v_0 to v_1 and v_2 respectively. Then $P_1 + e_1$ and $P_2 + e_2$ will be two different directed paths from v_0 to w, a contradiction. Hence every vertex of T, other than the root, must have in-degree 1.

Conversely, let v_0 be a vertex of a directed tree T with in-degree 0 and all other vertices of T have in-degree 1. Let u be an arbitrary vertex of T, other than v_0. Since u has in-degree 1, there is a unique vertex v_1 and an arc e_1 with tail v_1 and head u. Similarly, there is a unique vertex v_2 and an arc e_2 with tail v_2 and head v_1. Continuing the process we shall reach the vertex $v_k = v_0$ and the process stops here since v_0 has in-degree 0 and the vertices v_i do not repeat as the underlying graph of T has no cycle. Then $P = (v_0 = v_k, e_k, v_{k-1}, \ldots, v_1, e_1, u)$ will form a unique directed path from v_0 to u. Thus, T is a rooted tree with root v_0.

Corollary 5.3.1 The root of a rooted tree is unique.

We now give below a few definitions.

Level (or, depth) of a vertex : Let $T(v_0)$ be a rooted tree and u be a vertex of T. The **level or depth** of u is defined as the distance of u from the root v_0; i.e., the length of the unique path from v_0 to u. The depth of the root v_0 is 0.

Height of a rooted tree : Let $T(v_0)$ be a rooted tree. The **height** of T is defined as the length of a longest parth from the root, i.e., the greatest depth of T.

Parent and Child : Let $T(v_0)$ be a rooted tree. If a vertex u immediately precedes another vertex v on the path from the root v_0 to v, then u is called the **parent** of v and v is called a **child** (or offspring) of u.

Siblings : The children having the same parent are called **siblings**.

Descendant and Ancestor : Let $T(v_0)$ be a rooted tree and u, v be two distinct vertices of T. The vertex v is called a **descendant** of the vertex u if u is on the unique path from the root v_0 to v. In this case u is called an **ancestor** of v.

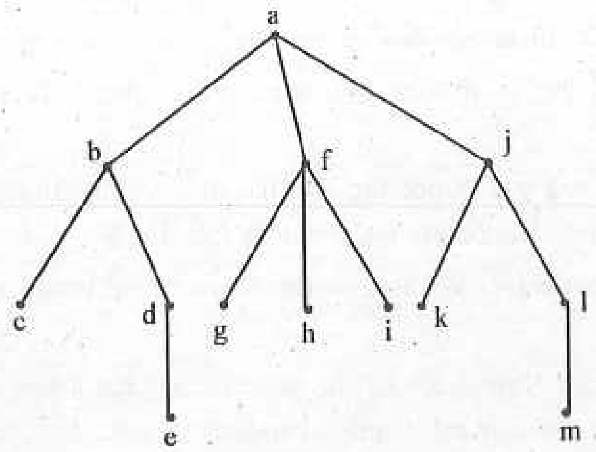
Subtree of a rooted tree : Let $T(v_0)$ be a rooted tree and $v \in V_T$, the vertex-set of T, other than v_0. Then T(v) is also a rooted tree with v as the root. We then say that T(v) is the **subtree** of $T(v_0)$ starting at v.

Leaf : Let $T(v_0)$ be a rooted tree. A vertex v of T is called a **leaf** if it has no child.

Internal Vertex : An **internal vertex** of a rooted tree is any vertex which is not a leaf.

Standard plane drawing of a rooted tree : Let $T(v_0)$ be a rooted tree. If we draw the graph T on a plane keeping v_0 at the top and the vertices at each level at the same horizontal line, then the drawing is called a **standard plane drawing** of the rooted tree.

Example 5.3.2 Consider the standard plane drawing of the following rooted tree in fig. 5.3.2.



Rooted Tree T(a)
Fig. 5.3.2

In the tree T, a is the root, the vertices b, f, j are in level 1; c, d, g, h, i, k, l are in level 2, and e, m are in level 3. The height of T is 3, a is the parent of the children b, f, j; b is the parent of the children c, d; f is the parent of the children g, h, i; j is the parent of the children k, l ; d is the parent of the child e and l is the parent of the child m. The children (b, f, j), (c, d), (g, h, i), (k, l) are siblings. The vertices c, d, e are descendants of b and b is their ancestor. Observe that there are other descendants and ancestors in the tree. The vertex a (the root) is the ancestor of all other vertices. The leaves of T are the vertices c, e, g, h, i, k, m. The internal vertices of T are a, b, f, j, d, l.

Ordered rooted tree : Let T be a rooted tree. T is called an **ordered tree** if the children of each parent in T are assigned a fixed ordering. Thus, if a parent u has three children, we may order them as the first, second or third child of u.

When we draw an ordered rooted tree on a plane we arrange the vertices at each level from left to right which agrees with the ordering of the children.

m-tree : Let T be a rooted tree. T is said to be an **m-tree** ($m \geq 1$) if every vertex of T has at most m children.

Complete m-tree : Let T be a rooted tree. T is said to be a **complete m-tree** if every internal vertex of T has exactly m children and all the leaves have the same level.

Theorem 5.3.2 *An m-tree has at most m^p vertices at level p.*

Proof : Let T be an m-tree. We shall prove the result by induction on the level p.

Basis step : Let p = 1. Since the root has at most m children, there are at most m vertices at level 1. Hence the statement is true for p = 1.

Induction hypothesis : We assume that there are m^k vertices at level k, for some $k \geq 1$.

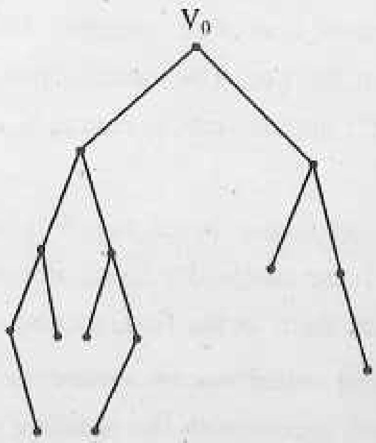
Induction step : Now, each of the vertices at level k has at most m children. Hence there are at most $m.m^k = m^{k+1}$ children at level k + 1. The theorem now follows by principle of induction.

5.4 Binary Tree

A **binary tree** is an ordered 2-tree in which every child is designated as a left child or a right child.

Complete binary tree : A binary tree T is called **complete** if every internal vertex of T has exactly 2 children, and all leaves have the same level.

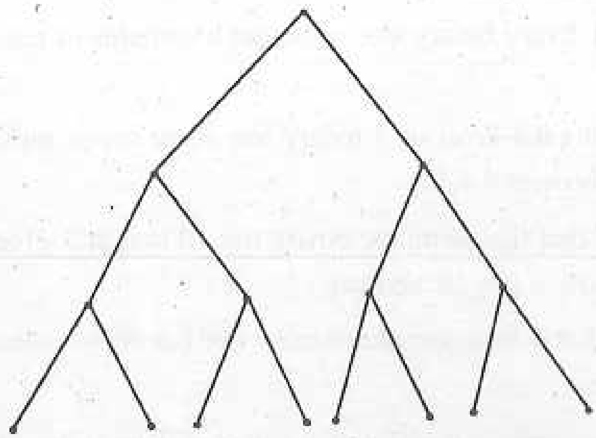
Example 5.4.1 The following graph is a binary tree.



A binary tree of height 4

Fig. 5.4.1

Example 5.4.2 The following graph is a complete binary tree.



A complete binary tree of height 3

Fig. 5.4.2

Left subtree and right subtree of a vertex : Let T be a binary tree and $v \varepsilon V_T$. The **left subtree** of v is the binary subtree with left child of v as the root and spanning all its descendants. Similarly, the **right subtree** of v is the binary subtree with right child of v as the root and spanning all its descendants.

Note : If v has no left (right) child, then its left (right) subtree is empty.

Theorem 5.4.1 *Let T be a complete binary tree of height h and with n vertices. Then,* $n = 2^{h+1} - 1$.

Proof : We prove the theorem by induction on h.

Basis Step : Let h = 0. Then T has only one vertex. Hence n = 1 and $1 = 2^{0+1} - 1 = 2 - 1 = 1$. Thus the result is true for h = 0.

Induction Hypothesis : We assume that for some $m \geq 0$, a complete binary tree of height m has $2^{m+1} - 1$ vertices.

Induction step : Let T be a complete binary tree of height m + 1. Let the root of T have the left child u_1 and the right child u_2. Let T_1 and T_2 be the left subtree and the right subtree with u_1 and u_2 as roots respectively. Since T is complete, T_1 and T_2 must also be complete. Also both the subtrees T_1 and T_2 are of height m. So, by induction hypothesis, they each contain $2^{m+1} - 1$ vertices. Hence the number of vertices of T is $2(2^{m+1} - 1) + 1 = 2^{m+2} - 1$. Thus, by the principle of induction, a complete binary tree T of height h contains $2^{h+1} - 1$ vertices; i.e., $n = 2^{h+1} - 1$ where n is the number of vertices in T.

Corollary 5.4.1 Every binary tree of height h contains at most $2^{h+1} - 1$ vertices; i.e., $n \leq 2^{h+1} - 1$.

Proof : Since at each level of a binary tree there are at most two children, the result follows from theorem 5.4.1.

Note : Observe that the complete binary tree of height 3 given in the Fig. 5.4.2 contains $2^{3+1} - 1 = 16 - 1 = 15$ vertices.

Theorem 5.4.2 Let T be a complete binary tree having p internal vertices. Then T contains p + 1 leaves.

Proof : T being complete, each internal vertex of T must have two children. Since there are p internal vertices, T has 2p children. Now, the root is the only vertex which is not a child of any vertex. Hence the total number of vertices in T is 2p + 1. Consequently, the number of leaves in T is $(2p + 1) - p = p + 1$.

Theorem 5.4.3 Let T be a binary tree of height h having k leaves. Then, $k \leq 2^h$.

Proof : We prove the theorem by induction on h.

Basis step : If h = 0, then T has only one vertex. Then $k = 1 = 2^0$. Thus the theorem holds when h = 0.

Induction Hypothesis : We assume that for some m such that $0 \leq m < h$, a binary tree of height m having k leaves, $k \leq 2^m$.

Induction Step : Let T be a binary tree of height m + 1 and having k leaves. We consider the following two cases.

Case 1. The root of T has only one child u.

Let T_1 be a subtree of T with u as root. The height of T_1 is m. Since the leaves of T_1 are those of T, T and T_1 have the same number of leaves. Hence T_1 has k leaves. So, by induction hypothesis, $k \leq 2^m < 2^{m+1}$

Case 2. The root of T has two children, say u_1 and u_2. Let T_1 and T_2 be the subtrees of T with u_1 and u_2 as roots respectively. Also, let k_1 and k_2 be the number of leaves of T_1 and T_2 respectively.

Then, $k = k_1 + k_2$ since T_1 and T_2 have no common leaves.

The height of both T_1 and T_2 is $\leq m$. Hence, by induction hypothesis, $k_1 \leq 2^m$ and $k_2 \leq 2^m$. Hence, $k = k_1 + k_2 \leq 2^m + 2^m = 2^{m+1}$.

Therefore, the theorem follows by the principle of induction.

5.5 Worked Out Exercises

1. Draw a graph having the given properties or, explain why no such graph can exist.

(a) A connected, acyclic graph with 7 vertices and 7 edges.

(b) A tree having degree-sequence (1, 3, 4, 4, 6)

(c) A tree with 13 vertices having four vertices of degree 3, three vertices of degree 4 and six vertices of degree 1.

(d) A binary tree with 4 internal vertices and 5 leaves.

(e) A binary tree of height 4 and 17 leaves.

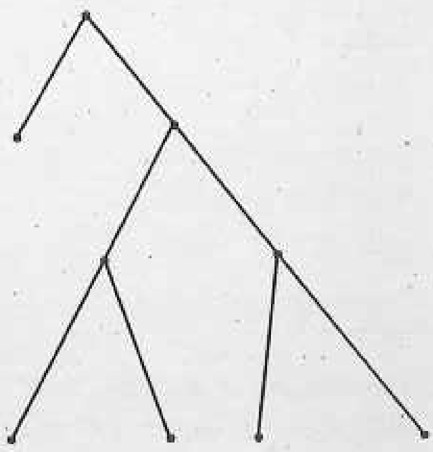
(f) A complete binary tree of height 3 and 12 vertices.

Solution : (a) Such a graph cannot exist. For, the graph is connected and acyclic implies it is a tree. It has 7 vertices. Hence it must have $7 - 1 = 6$ edges.

(b) Such a graph cannot exist. For, the graph is a tree and so it must have at least two vertices of degree 1.

(c) There can not be any such tree. Because, the sum of degrees of the vertices $= 4 \times 3 + 3 \times 4 + 6 \times 1 = 12 + 12 + 6 = 30$. Hence, number of edges must be 15.

(d) The required tree can be drawn as follows :



(e) No such graph can exist. For, the height h = 4. Then, number of leaves k must be $\leq 2^4$; i.e. $k \leq 16$.

(f) Such a tree does not exist. Because, height of the tree h = 3. Hence, number of vertices $n = 2^{3+1} - 1 = 15$.

2. Prove that a graph G is a forest if and only if $e - n + k = 0$, where e = number of edges of G, n = number of vertices of G and k = number of components of G.

Proof : Let G be a forest with n vertices, e edges and k components. Then, by theorem 5.1.6, $e = n - k$ i.e., $e - n + k = 0$.

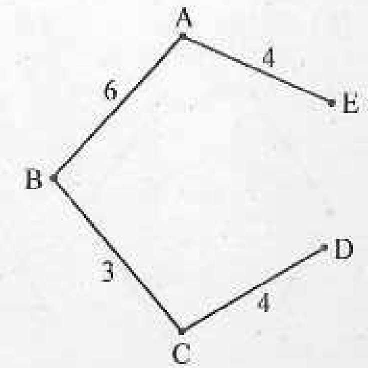
Conversely, let $e - n + k = 0$. Let $G_1, \ldots\ldots, G_k$ be the components of G with $n_1, \ldots\ldots, n_k$ vertices and $e_1, \ldots\ldots, e_k$ edges respectively. If possible, let at least one of the components of G, say G_i, contain a cycle. Since G_i is connected, $e_i \geq n_i - 1$. But G_i is not a tree and so $e_i \neq n_i - 1$. Hence, $e_i > n_i - 1$; i.e., $e_i \geq n_i$. Also, $e_j \geq n_j - 1$, $j = 1, \ldots\ldots, i-1, i+1, \ldots\ldots, k$. Hence $e = e_1 + \ldots\ldots + e_i + \ldots\ldots + e_k \geq (n_1 - 1) + \ldots\ldots + n_i + \ldots\ldots + (n_k - 1) = (n_1 + \ldots\ldots + n_k) - (k - 1) = n - k + 1 > n - k$. Hence, $e - n + k > 0$, a contradiction. Hence G_i does not contain a cycle; i.e., G_i is a tree. Hence G is a forest.

3. The following table shows the distances in km between five villages A, B, C, D, E. The villages are to be connected by a network of roads. Find a minimal spanning tree connecting the five villages applying Kruskal's algorithm.

	A	B	C	D	E
A	–	6	7	9	4
B	6	–	3	5	7
C	7	3	–	4	6
D	9	5	4	–	10
E	4	7	6	10	–

Solution : We denote the villages as the vertices of a weighted graph G, the roads connecting any two villages as the edges of G and the distance between any two villages as edge-weight. The weights in non-decreasing order are 3, 4, 4, 5, 6, 6, 7, 7, 9, 10. There are five vertices. So we must select $5 - 1 = 4$ edges for a minimal spanning tree of G.

The smallest weight is 3 between B and C. We select the edge joining B and C. The next smallest weight is 4 between either A and E or, between C and D. We can choose both the edges since they would not create any cycle. The next smallest weight is 5 between B and D. But we cannot choose an edge between B and D as this would create a cycle BCD. The next smallest weight is 6 between either A and B or between C and E. We can select an edge between any one of these pairs as none of them would create a cycle with the edges already selected. Let us choose an edge between A and B. These four chosen edges will form a minimal spanning tree of G of total weight 17 km.



4. Let p_1 denote the number of vertices of a tree T of degree 1 and q denote the number of vertices of degree ≥ 3. If T contains at least two vertices, prove that $p_1 \geq q + 2$. Also show that $p_1 = q + 2$ when T does not contain any vertex of degree > 3.

Proof : Let p_k denote the number of vertices of T having degree k. Then total number of vertices of T is $p_1 + p_2 + p_3 + ...$. Since number of edges in any graph $= \frac{1}{2}$ (Sum of the degrees of all the vertices), we have, number of edges in $T = \frac{1}{2}(p_1 + 2p_2 + 3p_3 + ...)$. Since T is a tree we get.

$$p_1 + p_2 + p_3 + ... = \frac{1}{2}(p_1 + 2p_2 + 3p_3 + ...) + 1.$$

That is, $2(p_1 + p_2 + p_3 + ...) = (p_1 + 2p_2 + 3p_3 + ...) + 2$.......... (1)

But obviously, $p_1 + 2p_2 + 3p_3 + 4p_4 + ... \geq p_1 + 2p_2 + 3(p_3 + p_4 + ...)$

Using (1) we get, $2(p_1 + p_2 + p_3 + ...) \geq p_1 + 2p_2 + 3(p_3 + p_4 + ...) + 2$

Hence, $p_1 \geq (p_3 + p_4 + ...) + 2 = q + 2$

The equality occurs, i.e., $p_1 = q + 2$ exactly if $p_4 = p_5 = ... = 0$.

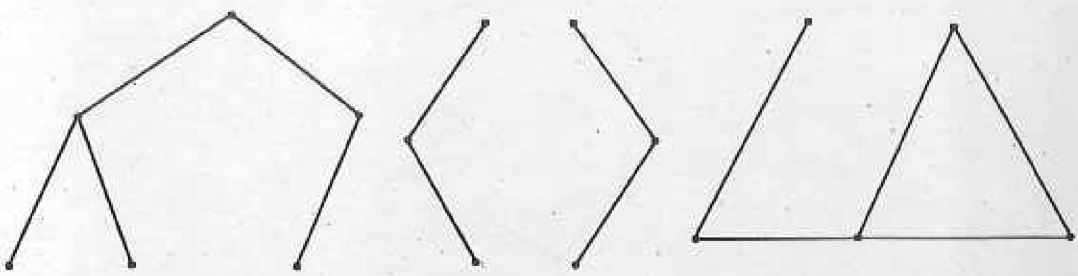
i.e., if T does not contain a vertax of degree > 3.

5.6 Exercises

1. Which of the following graphs are trees and which are not?

Justify your answer.

(a) **(b)** **(c)**



2. Let G be a connected graph with 8 vertices and 7 edges. Does G contain a vertex of degree 1? Justify your answer.

3. State whether the following statements are true or false. Justify your answer.

(a) A tree is a graph without cycles.

(b) A tree must be a simple graph.

(c) Every bipartite graph is a tree.

(d) Every tree with more than one vertices is a bipartite graph.

(e) If any two vertices of a graph G are connected by at least one path, then G must be a tree.

(f) Any graph with n vertices and n – 1 edges is a tree.

4. Does there exist a tree with 10 vertices such that the sum of the degrees of the vertices is 24? Justify your answer.

5. Draw a forest with 13 vertices, 9 edges and 4 components.

6. Draw a forest with 8 vertices and 5 edges.

7. Draw a graph having the given properties or, explain why no such graph can exist.

(a) Acyclic, four edges, six vertices.

(b) Tree, all vertices are even.

(c) Tree, 13 vertices with 9 vertices of degree 1, 3 vertices of degree 4, and one vertex of degree 3.

(d) Tree having degree-sequence (1, 1, 1, 1, 3, 3)

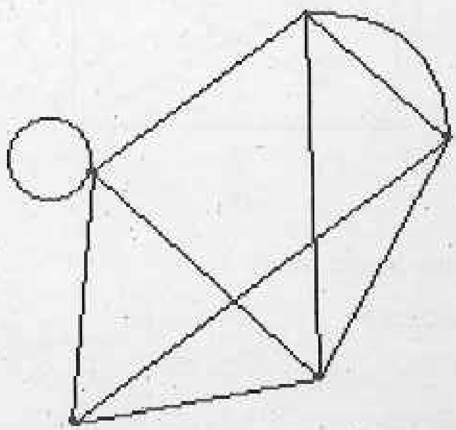
(e) A complete binary tree of height 4 and 31 vertices.

(f) A binary tree of height 5 and 64 vertices.

(g) A binary tree of height 3 and 10 leaves.

8. Prove that a connected graph with $n \geq 2$ vertices is a tree if and only if the sum of the degrees of the vertices is equal to $2(n - 1)$.

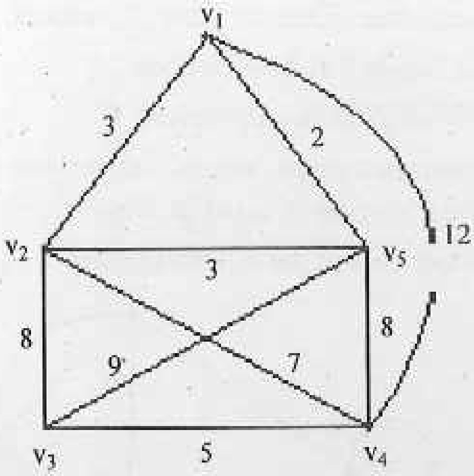
9. Draw two spanning trees of the following graph :



10. The following table shows the distances, in units of 10 km, between six cities A, B, C, D, E, F. The cities are to be connected by a network of telephone lines. Find a minimal spanning tree connecting the six cities applying Kruskal's algorithm.

	A	B	C	D	E	F
A	–	4	6	10	4	8
B	4	–	12	3	2	5
C	6	12	–	8	6	6
D	10	3	8	–	7	9
E	4	2	6	7	–	8
F	8	5	6	9	8	–

11. For the following connected weighted graph G, find a minimal spanning tree of G applying Kruskal's algorithm :



G

12. What is the maximum height for :

(a) A tree on the vertex-set $S = [v_1, \ldots\ldots, v_6]$?

(b) A complete binary tree on the vertex-set S = {a, b, c, d, e, f, g}?

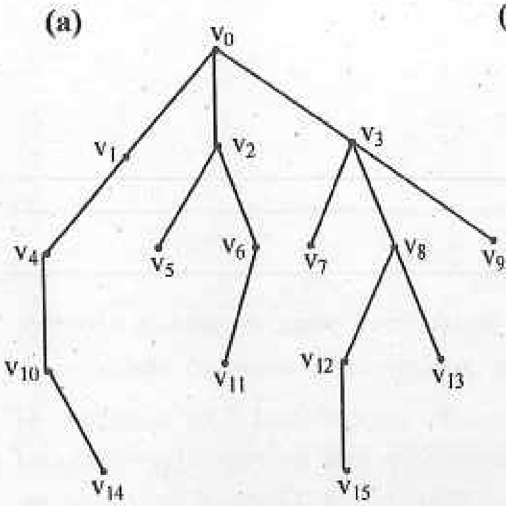
(c) A binary tree on the vertex-set $S = \{v_1, \ldots\ldots, v_5\}$?

Justify your answer in each case.

13. For the rooted trees given below, find the following : (i) Root, (ii) Height, (iii) Internal vertices, (iv) Leaves, (v) Vertices in the same levels, (vi) Parents and children, (vii) Siblings, (viii) Descendants of parents.

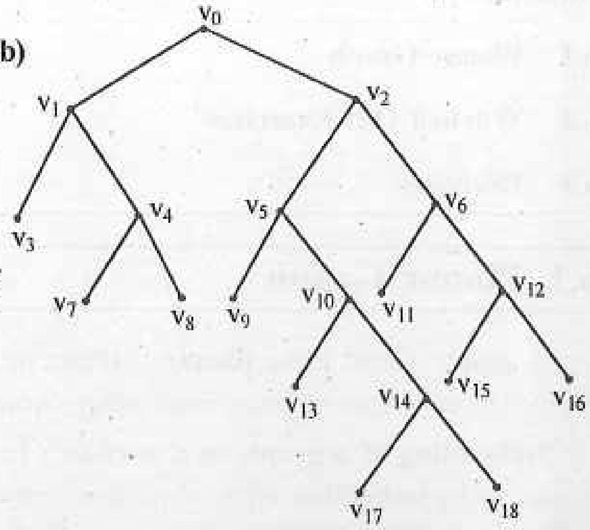
State also the types of the trees.

(a)



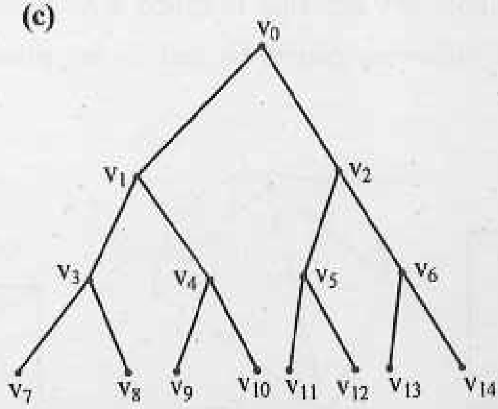
T_1

(b)



T_2

(c)



T_3

14. Let T be a tree with an even number of edges. Prove that T must contain at least one vertex having even degree.

Solution : (Hints) Let T contain m (even) edges. Since T is tree, number of vertices in T = m + 1 (odd). If all the vertices are of odd degree, the sum of the degrees of the vertices will be odd, a contradiction.

Unit 6 □ Planar Graph

Structure

6.1 Planar Graph

6.2 Worked Out Exercises

6.3 Exercises

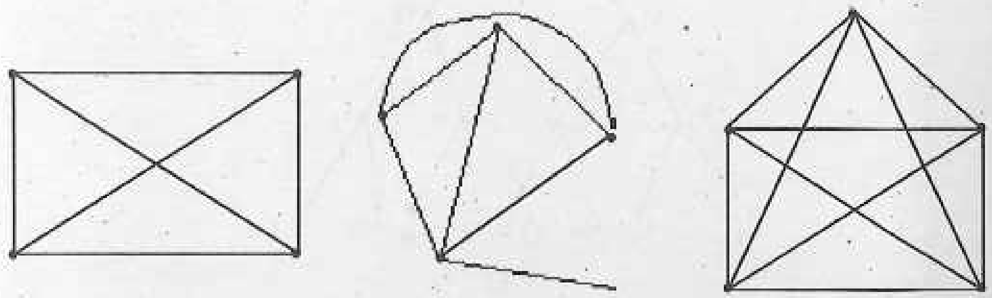
6.1 Planar Graph

A graph is said to be **planar** if it can be drawn on a plane in such a way that no two of its edges intersect each other except possibly at a common vertex.

Imbedding of a graph on a surface : Let G be a graph and S be a surface. G is said to be **imbedded** on S when it is drawn on S so that no two edges intersect except possibly at a common vertex. Such a drawing of G on S is called an imbedding of G on S.

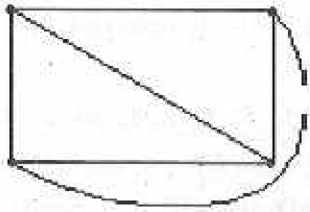
Thus, a graph is planar if and only if it can be imbedded on a plane. A drawing of a planar graph G without any crossing is called a plane representation of G.

Example 6.1.1 The following graphs G_1 and G_2 are planar, whereas the graph K_5 is non-planar.



G_1 G_2 K_5

The graph G_2 is clearly planar since it has been drawn on the plane such that none of its edges cross each other. The graph G_1 is also planar because we can redraw it on the plane in a different way so that none of its edges cross. For example, we can redraw G_1 as follows :



G_1

However, the graph K_5 is not planar. We shall prove it later. Just looking at the drawing of a particular graph it may not be possible to say whether the graph is planar or not. At the first glance G_1 seems to be non-planar since two edges cross each other. However, it is possible to redraw the graph so that no two edges cross each other. A redrawing of a graph without crossings is not always possible.

Faces of a planar graph : Let G be a planar graph. Then any plane representation of G divides the plane into regions. The union of a region and its boundary is called a *face* of G. We always consider the outer region, i.e., unbounded region with its boundary to be a face, called exterior face.

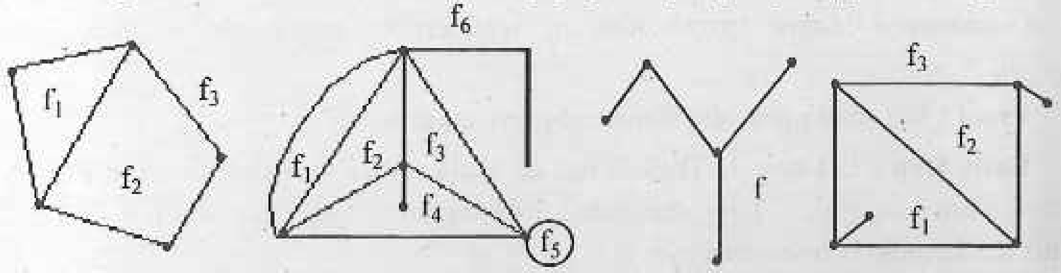
Boundary walk of a face : The **boundary walk** of a face f of a planar graph G is a closed walk in G covering the perimeter of the face. Observe that vertices and edges may be repeated in a boundary walk.

Size (or, degree) of a face : Let f be a face of a planar graph G.

The **size** of f is defined as the number of edge-steps needed in the boundary walk of f.

A face of size n is said to be an **n-sided face.** Observe that the number of edge-steps in a boundary walk of a face may be more than the number of edges on the face-boundary, because some edges may be repeated in the boundary walk. Hence the size of a face f may be greater than the number of edges on the boundary of f.

Example 6.1.2 Consider the following graphs G_1, G_2, G_3, G_4 given in fig. 6.1.2.



G_1 G_2 G_1 G_4

Fig. 6.1.2

In G_1, there are three faces f_1, f_2, f_3 (exterior face). Size $(f_1) = 3$, size $(f_2) = 4$, size $(f_3) = 5$.

In G_2, there are six faces f_1, f_2, f_3, f_4, f_5, f_6 (exterior face).

Size $(f_1) = 2$, size $(f_2) = 3$, size $(f_3) = 3$, size $(f_4) = 5$ (Here one edge-step must be repeated as the walk is closed), size $(f_5) = 1$, size $(f_6) = 8$.

G_3 is a tree and there is only one face, the exterior face f. In fact, a tree has only one face since it is acyclic. Here size (f) = 8. Observe that for a tree with m edges with the exterior face f, the size (f) = 2xm, since in the boundary walk of f, every edge must be repeated once.

In G_4, there are three faces f_1, f_2, f_3 (exterior face), size $(f_1) = 5$, size $(f_2) = 3$, size $(f_3) = 6$.

Face-size Equation : Theorem 6.1.1 *Let G be a planar graph, n_e = number of edges in G and F = the set of all faces of G. Then,* $2n_e - \sum_{f \varepsilon F} size(f)$.

Proof : If an edge is a common border of two different faces, then the edge will be counted once in each of the boundary walks of the faces so that it will contribute 2 to the sum of the face-sizes. If the edge is not a common border of two different faces, then it occurs twice in the same boundary walk. Thus, every edge of G contributes 2 to the sum of the face-sizes. Hence, $2n_e = \sum_{f \varepsilon F} size(f)$.

Note : Observe that in the graph G_4 shown in the fig. 6.1.2, $n_e = 7$ and $\sum_{f \varepsilon F} size(f) = 5 + 3 + 6 = 14$ so that $2n_e = \sum_{f \varepsilon F} size(f)$.

This is also true for the other graphs G_1, G_2, G_3.

Euler's Formula (or, Euler polyhedral equation) : Theorem 6.1.2. *If G is a connected planar graph with n_v vertices, n_e edges and n_f faces, then $n_v - n_e + n_f = 2$.*

Proof : We shall prove the theorem by induction on n_f.

Basis Step : Let $n_f = 1$. Then G has no cycle. Since G is connected, it must be a tree. Hence, $n_e = n_v - 1$ (cf. theorem 5.1.4) and so $n_v - n_e + n_f = n_v - n_v + 1 + 1 = 2$. Thus the formula is true when $n_f = 1$.

Induction Hypothesis : We assume that the formula holds for any connected planar graph with $n_f < k$, $k \geq 2$, k an integer.

Induction Step : Let G be any connected planar graph with n_v vertices, n_e edges and $n_f = k$ faces. Since $k \geq 2$, G is not a tree and hence must have at least one cycle. We choose an edge e which is a cycle-edge. Then G – e is a connected planar graph with n_v vertices, $n_e - 1$ edges and $k - 1$ faces, since the two faces of G separated by e combine together to form one face of G – e.

Therefore, for the graph G – e, by induction hypothesis,

$n_v - (n_e - 1) + (n_f - 1) = 2$, that is, $n_v - n_e + n_f = 2$.

The theorem now follows by the principle of mathematical induction.

Theorem 6.1.3 *In any simple connected planar graph with n_v vertices, n_e (>2) edges and n_f faces, the following inequalities hold :*

(i) $2n_e \geq 3n_f$, (ii) $n_e \leq 3n_v - 6$

Proof : (i) Let G be a simple connected planar graph. Since G is simple, each cycle (if any) has at least three edges. Hence each face is bounded by at least three edges. Then the total number of occurrences of the edges that bound the faces is at least $3n_f$. Again, each edge either occurs once in each of two different face boundary walks or occurs twice in the same boundary walk.

Hence, $2n_e \geq 3n_f$.

(ii) From Euler's formula we have, $n_e = n_v + n_f - 2$.

So, from (i) we get $n_e \leq n_v + \frac{2}{3}n_e - 2$; i.e. $\frac{1}{3}n_e \leq n_v - 2$.

i.e., $n_e \leq 3n_v - 6$

Corollary : 6.1.1 Let G be a simple connected graph such that $n_e > 3n_v - 6$. Then, G is non-planar.

Corollary : 6.1.2 Let G be a simple connected graph such that $2n_e < 3n_f$. Then, G is non-planar.

Theorem 6.1.4 *Let G be a connected bipartite planar graph with n_v vertices and n_e edges. Then, $n_e \leq 2n_v - 4$.*

Proof : Since G is a connected bipartite graph, each cycle (if any) in G has at least four edges. Hence each face is bounded by at least four edges. Then total number of occurrences of the edges that bound the faces is at least $4n_f$. Again, each edge either occurs once in each of two different face boundary walks or occurs twice in the same boundary walk. Hence, $2n_e \geq 4n_f$; i.e., $n_e \geq 2n_f$.

From Euler's formula we have,

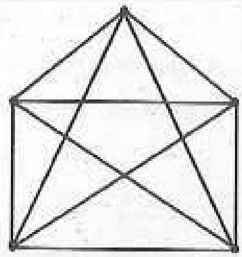
$n_v - n_e + \frac{1}{2}n_e \geq 2$; or, $n_v - \frac{1}{2}n_e \geq 2$ i.e., $n_e \leq 2n_v - 4$.

Corollary : 6.1.3 Let G be any connected bipartite graph such that $n_e > 2n_v - 4$. Then G is non-planar.

Theorem 6.1.5 *The graphs (i) K_5 and (ii) $K_{3,3}$ are non-planar.*

Proof : (i) The complete graph K_5 is a simple connected graph.

The graph K_5 is shown below.

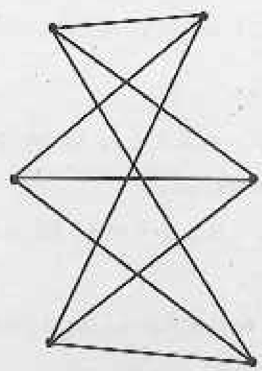


K_5

For K_5, $n_e = 5_{C_2} = 10$, $n_v = 5$. Then, $3n_v - 6 = 15 - 6 = 9$ and $10 = n_e > 3n_v - 6$.

Hence, by corollary 6.1.1., K_5 is not a planar graph.

(ii) The complete bipartite graph $K_{3,3}$ is a simple connected graph. The graph $K_{3,3}$ is shown below :



$K_{3,3}$

For $K_{3,3}$, $n_v = 6$, $n_e = 3 \times 3 = 9$.

Then, $2n_v - 4 = 12 - 4 = 8$.

Hence, $9 = n_e > 2n_v - 4$. Hence, by corollary 6.1.3, $K_{3,3}$ is not a planar graph.

Remark 6.1.1 The two graphs K_5 and $K_{3,3}$ are known as Kuratowski's graphs. K_5 is Kuratowski's first graph and $K_{3,3}$ is Kuratowski's second graph.

K_5 and $K_{3,3}$ have some common properties apart from the fact that they are both non-planar. These common properties are as follows :

(i) They are both regular graphs.

(ii) If we remove one vertex or an edge from the graphs, each of them becomes a planar graph.

(iii) Both of them are simplest non-planar graphs in the sense that K_5 is the non-planar graph with the smallest number of vertices and $K_{3,3}$ is the non-planar graph with the smallest number of edges.

Subdivision of an edge of a graph : Let G be a graph and e be an edge of G with end points {u, v}. **Subdivision** of the edge e is an operation that inserts a new vertex, say w, on the edge between u and v and as a result the edge e is split into two edges e_1 and e_2 say, with end points {u, w} and {w, v} respectively.

The subdivision of an edge e of a graph is shown below :

The edge e   Subdivision of e

Subdivision of a graph : If we perform a finite sequence of edge-subdivision operations on a graph G, then the resulting graph is called a **subdivision of the graph G.**

Graph Homeomorphism : Two graphs G and H are said to be **homeomorphic** if there is an isomorphism between a subdivision of G and a subdivision of H.

Example 6.1.3 Consider the following two graphs G and H as shown below in fig. 6.1.3.

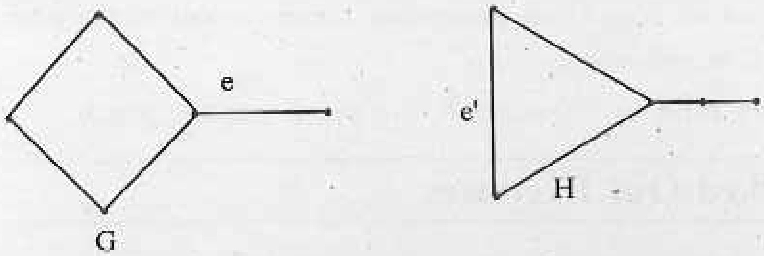


Fig. 6.1.3

The graphs G and H cannot be isomorphic since G contains a cycle of length 4 whereas H contains a cycle of length 3. But G and H are homeomorphic, because if the edge e in G and the edge e' in H are both subdivided, then the resulting subdivisions G_1 of G and H_1 of H are isomorphic. The subdivisions G_1 and H_1 are shown below in fig. 6.1.4.

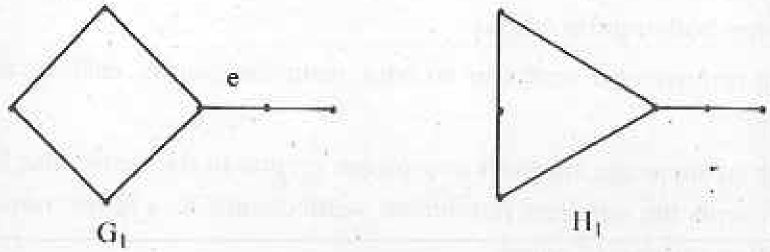


Fig. 6.1.4

We have shown that Kuratowski's graphs K_5 and $K_{3,3}$ are non-planar.

For a long time it was an unsolved problem to characterize planar graphs. In 1930 the Polish mathematician Kazimierz Kuratowski (1896-1980) solved this problem. He proved a necessary and sufficient condition for the planarity of a graph.

Theorem 6.1.6 (Kuratowski) : *A graph is planar if and only if it does not contain any subgraph homeomorphic to K_5 or, $K_{3,3}$.*

Alternative Statement of Kuratowski's theorem : A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Proof : There are several proofs of this theorem. We omit the proof since it is too involved. However, a proof can be found in [2, p109-112] or in [3. p314-320].

Theorem 6.1.7 *If a graph G contains a non-planar subgraph, then G is non-planar.*

Proof : If possible, let G be a planar graph and G_1 be a subgraph of G. Then G_1 is obtained from G by deleting some edges and vertices of G. Since G is planar, every plane drawing of G remains free of edge-crossings when we delete any set of edges and vertices of G. Thus G_1 is also planar which contradicts the given condition.

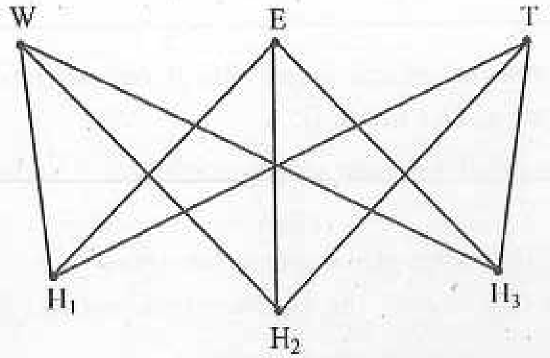
Hence, G is non-planar.

Corollary 6.1.4 Every subgraph of a planar graph is planar.

6.3 Worked Out Exercises

1. **Three Utilities Problem :** There are three houses each of which are to be connected to three utility service centres : water, electricity, telephone. In order to maintain the pipelines independently without affecting the others, the pipes must be laid on a plane surface so that they do not cross each other. Can you find a layout of the pipes so that none of the nine pipes cross each other?

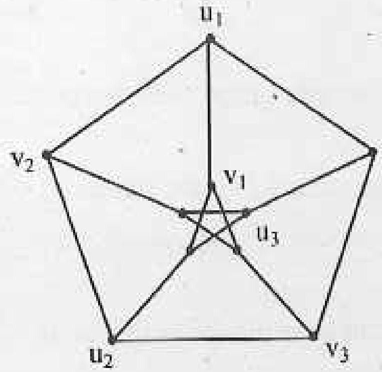
Solution : We take the three houses and the three utility service centres as the vertices of a graph, and the pipes connecting a service centre with a house as an edge of the graph. Then the problesm can be represented by the following graph :



The graph is $K_{3,3}$. The problem then reduces to the question : Is $K_{3,3}$ planar? Since $K_{3,3}$ is non-planar, it is impossible to find such a layout of the pipes.

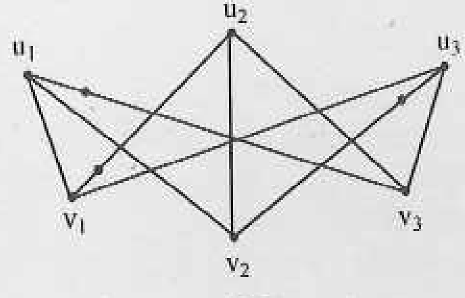
2. Using Kuratowski's theorem show that Petersen graph is non-planar.

Solution : Petersen graph (Danish mathematician Julius Petersen (1839-1910) is shown below :



Petersen graph

We now draw a subdivision of $K_{3,3}$ as follows :



We see that Petersen graph contains a subdivision of $K_{3,3}$.

Hence, by Kuratowski's theorem, Petersen graph is non-planar.

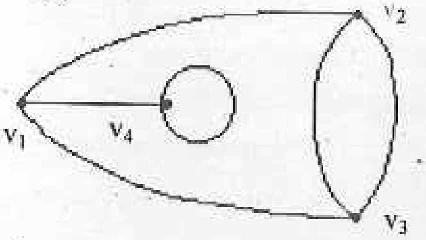
6.4 Exercises

1. Let G be a connected planar graph with 9 vertices having degree sequence (2,2,2,3,3,3,4,4,5). Find n_e and n_f for G.

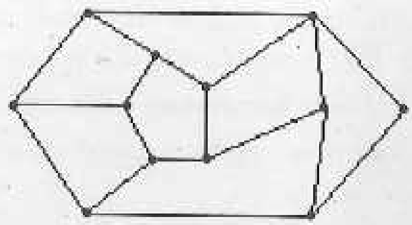
2. Let G be a connected 4-regular planar graph with 8 vertices. How many faces does G have?

3. For each of the following planar graphs how many faces does the graph have? Find the size of each face. Verify the face-size equation and Euler's formula.

(a)



(b)



4. Let G be a connected simple graph with 8 vertices and more than 18 edges. Prove that G is non-planar.

5. Show that the complete graph K_6 is non-planar.

6. Let G be a simple connected planar graph. Prove that there exists a vertex $v \in v_G$ such that $d(v) \leq 5$.

7. Let G be a simple connected planar graph with n_v vertices, n_e edges and n_f faces. If every face of G is bounded by m edges, show that $n_e = \frac{m(n_v - 2)}{m-2}$.

Unit 7 □ Matrix Representation of Graphs

Structure

7.1 Matrix Representation of a Graph

7.2 Exercises

7.3 References

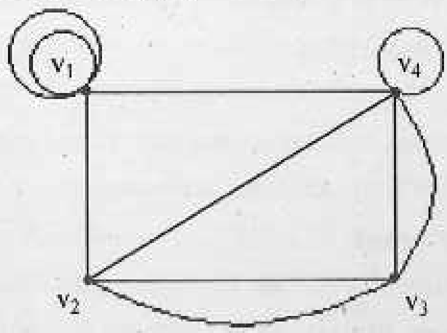
7.1 Matrix Representation of a Graph

The geometrical representation of a graph has certain disadvantages. They are useful only when number of vertices and edges are relatively small. But such representations become very cumbersome and sometimes impossible if the number of vertices and edges are very large. We now give an alternative method of representing a graph by means of a matrix. This method has the advantage in the sense that matrices can be stored in a computer memory easily and can be manipulated. The operations of matrix algebra can be used to find different characteristics of a graph.

Adjacency Matrix of a graph : Let G be a graph with n vertices and suppose that the n vertices are ordered as $v_1, v_2, \ldots\ldots, v_n$. Then the **adjacency matrix** of G, denoted A_G, with respect to the given ordering of V_G is an $(n \times n)$ matrix $[a_{ij}]$ such that

$$a_{ij} = \begin{cases} \text{the number of edges joining } v_i \text{ and } v_j \text{ when } i \neq j. \\ \text{the number of loops at } v_i \text{ if } i = j \end{cases}$$

Example 7.1.1 Consider the graph as shown below in fig. 7.1.1



G

Fig. 7.1.1

Suppose the vertex-set $V_G = \{v_1, v_2, v_3, v_4\}$ has been ordered as (v_1, v_2, v_3, v_4). Then the adjacency matrix A_G with respect to the given ordering of V_G is given by the following 4×4 matrix :

$$A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix} \end{array}$$

If we take the ordering (v_2, v_4, v_1, v_3), of V_G, then the corresponding adjacency matrix A_G is given by the following 4×4 matrix :

$$A_G = \begin{array}{c} \\ v_2 \\ v_4 \\ v_1 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccc} v_2 & v_4 & v_1 & v_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix} \end{array}$$

Note : When the ordering of the vertex-set V_G is implicit from the context, we need not label the rows and the columns and the adjacency matrix A_G can be written as a matrix without any row or coloumn labels.

Remark 7.1.1 When the adjacency matrix A_G is given we can draw the corresponding graph :

The matrix A_G is symmetric about the main diagonal. This is true for any adjacency matrix, because if v_i is joined to v_j by an edge, then v_j is also joined to v_i by the same edge. Although we get different adjacency matrices for different orderings of the vertex-set, the corresponding graphs will be the same.

Remark 7.1.2 Two graphs will be isomorphic if we can order their respective vertex-sets so that their adjacency matrices are identical.

Incidence Matrix of a graph : Let G be a graph with n vertices and m edges. Suppose that the vertices are ordened as v_1..........,v_n and the edges are ordered as e_1,....,e_m. Then the **incidence matrix** of G, denoted I_G, with respect to the given orderings of V_G and E_G, is an $(n \times m)$ matrix $[I_{ij}]$ in which

$$I_{ij} = \begin{cases} 0, \text{if } v_i \text{ is not an end point of } e_j, \\ 1, \text{ if } v_i \text{ is an end point of } e_j, e_j \text{ being not a loop,} \\ 2, \text{ if } e_j \text{ is a loop at } v_i. \end{cases}$$

Example 7.1.2 Consider the graph as shown below in fig. 7.1.2

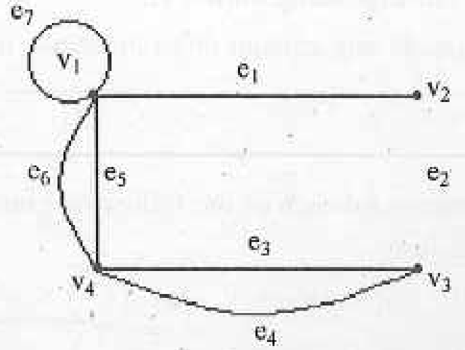


Fig. 7.1.2

Suppose the vertex-set V_G has been ordered as (v_1, v_2, v_3, v_4) and the edge-set E_G has been ordered as $(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$. Then the incidence matrix I_G with respect to the given orderings of V_G and E_G is given by the following 4×6 matrix :

$$I_G = \begin{matrix} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

If we take a different ordering of V_G, say, (v_2, v_4, v_1, v_3) and a different ordering of E_G, say, $(e_1, e_3, e_5, e_7, e_2, e_4, e_6)$ then the corresponding incidence matrix I_G will be given by the following 4×6 matrix :

$$I_G = \begin{matrix} & \begin{matrix} e_1 & e_3 & e_5 & e_7 & e_2 & e_4 & e_6 \end{matrix} \\ \begin{matrix} v_2 \\ v_4 \\ v_1 \\ v_3 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

Note : Observe that incidence matrix I_G also depends on the particular orderings of V_G and E_G. However, when the orderings of the vertex-set and the edge-set are implicit from the context, we need not label the rows and the columns and I_G can be written as a matrix without any row or column labels.

Remark 7.1.3 If we change the orderings of V_G and E_G, then the rows and columns of I_G are simply permuted.

From the definition of incidence matrix it follows that :

(i) The sum of the entries in any row, say i th row, of an incidence matrix I_G is equal to the degree of the corresponding vertex v_i.

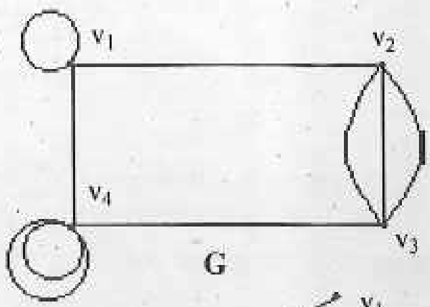
(ii) The sum of the entries in any column of an incidence matrix I_G is equal to 2.

7.2 Exercises

1. Find the adjacency matrix for each of the following graphs with respect to the given ordering of the vertex-set.

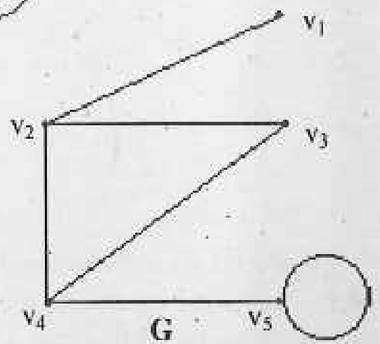
(a) Ordering of V_G– (i) (v_1, v_2, v_3, v_4)
(ii) (v_1, v_4, v_3, v_2)

(a)



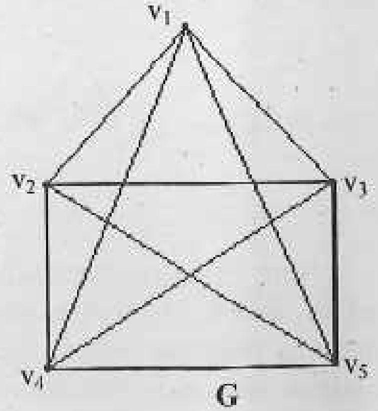
(b) Ordering of V_G– (i) $(v_1, v_2, v_3, v_4, v_5)$
(ii) $(v_1, v_2, v_4, v_5, v_3)$

(b)



(c) Ordering of V_G– (i) $(v_1, v_2, v_3, v_4, v_5)$
(ii) $(v_1, v_5, v_2, v_4, v_3)$

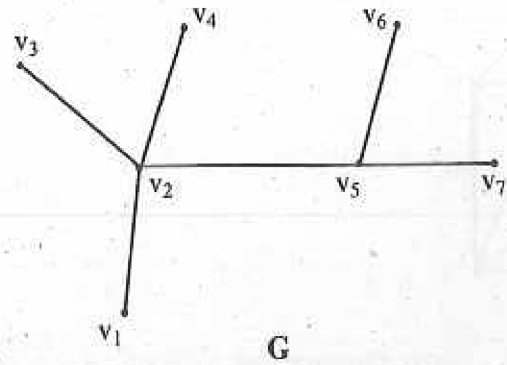
(c)



(d) Ordering of V_G– (i) $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$

(ii) $(v_7, v_6, v_5, v_4, v_3, v_2, v_1)$

(d)



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2. Find the incidence matrix for each of the following graphs with respect to the given orderings of the vertex-set and the edge-set.

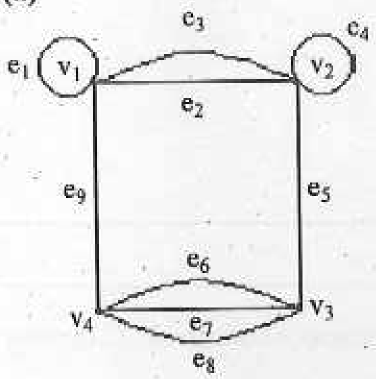
(a) Ordering of V_G– (i) (v_1, v_2, v_3, v_4)

Ordering of E_G– (i) $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9)$

Ordering of V_G– (ii) (v_1, v_4, v_3, v_2)

Ordering of E_G– (ii) $(e_1, e_3, e_5, e_7, e_9, e_2, e_4, e_6, e_8)$

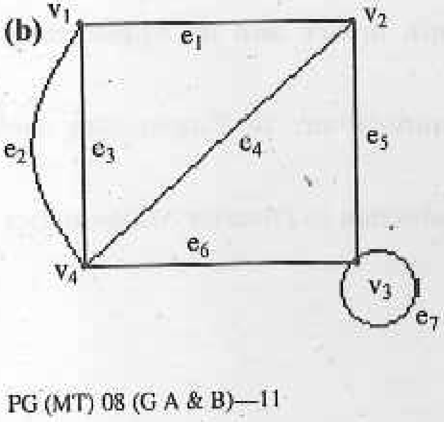
(a)



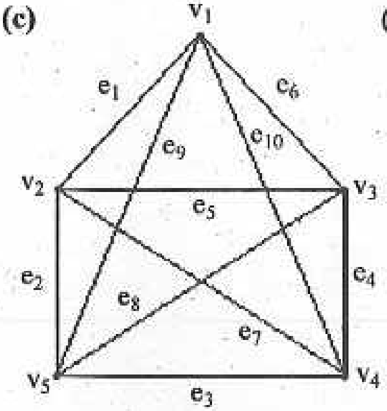
(b) Ordering of V_G– (v_1, v_2, v_3, v_4)

Ordering of E_G– $(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$

(b)



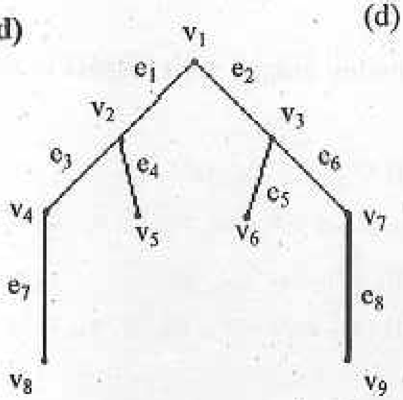
(c)



(c) Ordering of V_G– (v_1, v_2, v_3, v_4, v_5)

Ordering of E_G– (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10})

(d)



(d) Ordering of V_G– (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)

Ordering of E_G–(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)

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NOTES

NOTES

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Published by Netaji Subhas Open University, DD-26, Sector-I, Salt Lake, Kolkata-700064 & Printed at Gita Printers, 51A, Jhamapukur Lane, Kolkata-700009.